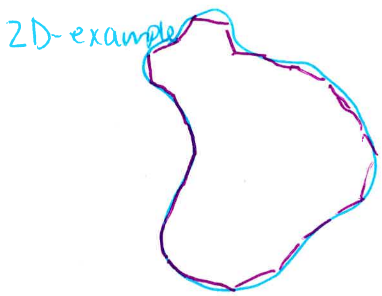
Math 8 - Lecture 17 — Nadia Lafrenière, 02/12/2020

Arc length, curvature

Length of a curve

Let $\vec{r(t)}$ be a curve defined by $\vec{r(t)} = \langle f(t), g(t), h(t)\rangle$.

What is the length of the curve from $t=a$ to $t=b$?

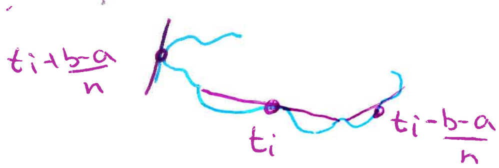


We can approximate the curves by line segments, and then compute their sum:

If we divide the curve into n many segments, the i^{th} has length (in 2D)

$$\underbrace{\frac{b-a}{n}}_{\Delta t}\left(\sqrt{f'(t_i)^2+g'(t_i)^2}\right),$$ where t_i is in the i-th segment.

why?



Taking the limit of that sum, as $n \to \infty$, we get

$$\int_a^b \sqrt{f'(t)^2+g'(t)^2}\,dt,$$ to express the length of the curve between a and b.

In three dimensions, the length of the curve $\vec{r(t)}$ from $t=a$ to $t=b$ is

$$\int_a^b \sqrt{f'(t)^2+g'(t)^2+h'(t)^2}\,dt$$

Example

The length of the helix $\langle \cos(t), \sin(t), t\rangle$ between $t=0$ and $t=2\pi$ is

$$\int_0^{2\pi} \sqrt{\sin^2(t)+\cos^2(t)+1}\,dt = \int_0^{2\pi}\sqrt{2}\,dt = \left(\sqrt{2}t\right|_0^{2\pi} = 2\sqrt{2}\pi.$$

Example

Find the length of the curve $\langle \frac{t^5}{5}, \frac{t^4}{2}, \frac{2t^3}{3} \rangle$ for t between 0 and 1.

Solution:

Then, $\vec{r'(t)} = \langle t^4, 2t^3, 2t^2 \rangle$.

The arclength is given by the integral

$$\int_0^1 \sqrt{t^8 + 4t^6 + 4t^4}\, dt = \int_0^1 \sqrt{t^4(t^4 + 4t^2 + 4)}\, dt$$

$$= \int_0^1 t^2 \, |t^2 + 2| \, dt$$

$$= \int_0^1 t^4 + 2t^2 \, dt$$

$$= \left(\frac{t^5}{5} + \frac{2t^3}{3} \right) \Big|_0^1$$

$$= \frac{1}{5} + \frac{2}{3} = \frac{13}{15}$$

That curve is $\frac{13}{15}$ units long.

Definition

The curvature of a smooth curve (i.e. a differentiable curve with non-zero vector derivative) measures how quickly the curve changes direction at a point.

A curve is smooth if it has no sharp corners or cusps.

For the curve defined by $\vec{r(t)}$, the unit tangent vector at point t is

$$\vec{T(t)} = \frac{\vec{r'(t)}}{|\vec{r'(t)}|}.$$

The curvature of $\vec{r(t)}$ is computed by

$$\kappa(t) = \frac{|\vec{T'(t)}|}{|\vec{r'(t)}|}$$

Greek letter kappa (κ)

Example

A circle of radius a has curvature $\frac{1}{a}$ everywhere:

$$\vec{r(t)} = \langle a\cos(t), a\sin(t)\rangle, \quad t \in [0, 2\pi).$$

$$\vec{r'(t)} = \langle -a\sin(t), a\cos(t)\rangle = a\vec{T(t)}$$ (because $\vec{r'(t)}$ is a vector of length a)

So, $\vec{T'(t)} = \left(\frac{\vec{r'(t)}}{a}\right)' = \langle -\cos(t), -\sin(t)\rangle$.

Hence, the curvature of the circle is

$$\kappa(t) = \frac{|\langle -\cos(t), -\sin(t)\rangle|}{|\langle -a\sin(t), a\cos(t)\rangle|} = \frac{1}{a}.$$

What does that mean? That the smallest a circle is, the more curved it is locally.

Theorem

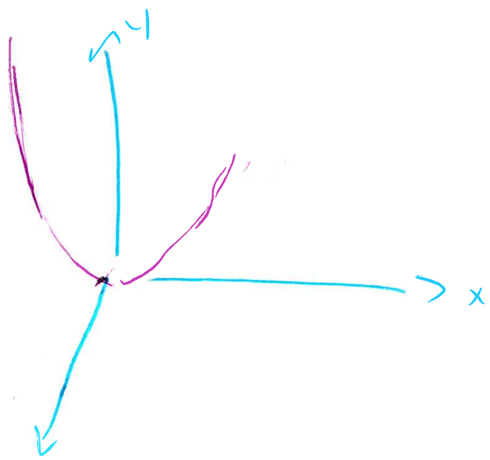
The curvature of the curve given by the function $\vec{r(t)}$ is

$$\kappa(t) = \frac{|\vec{r'(t)} \times \vec{r''(t)}|}{|\vec{r'(t)}|^3}$$

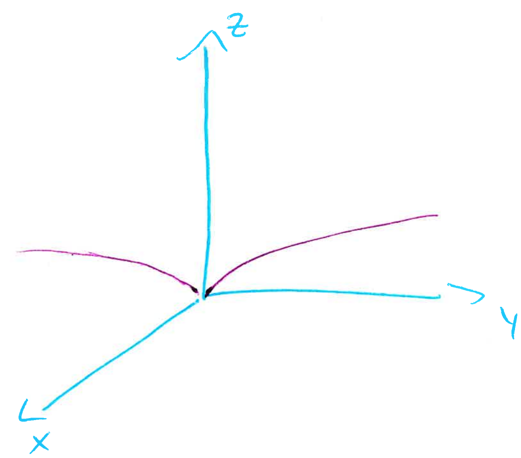
Example

Find the curvature of the twisted curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at a general position, and at $t=0$.

Where should it be maximal?



but also



It is still smooth (think of a lace with a loose loop), but the curvature should be maximal close to $t=0$.

To compute it:

$$\vec{r'}(t) = \langle 1, 2t, 3t^2 \rangle$$

and $$\vec{r''}(t) = \langle 0, 2, 6t \rangle.$$

So, $$\vec{r'}(t) \times \vec{r''}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \vec{i} \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}$$

$$= 6t^2 \vec{i} - 6t \vec{j} + 2\vec{k}$$

The theorem says that

$$\kappa(t) = \frac{|\langle 6t^2, -6t, 2 \rangle|}{|\langle 1, 2t, 3t^2 \rangle|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{\left(\sqrt{1 + 4t^2 + 9t^4}\right)^3}$$

At $t=0$, the curvature is 2.

If you want to check that the curvature is maximal close to 0, check that $\kappa'(0) = 0$.

Example

Find the curvature of the parabola $y=x^2$ at $(0,0)$, $(1,1)$ and $(2,4)$. What should the limit as $x \to \infty$ be?

This curve is defined by $\vec{r}(t) = \langle t, t^2 \rangle$ for $t \in \mathbb{R}$.

Also,

$$\vec{r}'(t) = \langle 1, 2t \rangle = \vec{i} + 2t\vec{j}$$
$$\vec{r}''(t) = \langle 0, 2 \rangle = 2\vec{j}.$$

Embedding in $\mathbb{R}^3$ to be able to compute the cross product. Simply imagine $\vec{r}(t) = \langle t, t^2, c \rangle$ for $c \in \mathbb{R}$.

Hence,

$$\vec{r}'(t) \times \vec{r}''(t) = (\vec{i} + 2t\vec{j}) \times 2\vec{j}$$
$$= \vec{i} \times 2\vec{j} + \cancel{2t\vec{j} \times 2\vec{j}}^{\,0} \text{ (parallel vectors)}$$
$$= 2\vec{k},$$

and $|\vec{r}'(t) \times \vec{r}''(t)| = 2.$

The length of $\vec{r}'(t)$ is $\sqrt{1+4t^2}$, which gives the curvature:

$$K(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{2}{(1+4t^2)^{3/2}}.$$

At $(0,0)$, the curvature is 2.

At $(1,1)$, it is $\dfrac{2}{\sqrt[2]{5^3}} \approx 0.179.$

At $(2,4)$, it is $\dfrac{2}{(17)^{3/2}} \approx 0.029$

It seems to be decreasing...

$$\lim_{t \to \infty} K(t) = \lim_{t \to \infty} \frac{2}{(1+4t^2)^{3/2}} = 0.$$

That means that, as $t \to \infty$, the parabola resembles a straight line.

Reference: James STEWART. Calculus, 8th edition, § 13.3