Math 8 - Lecture 18

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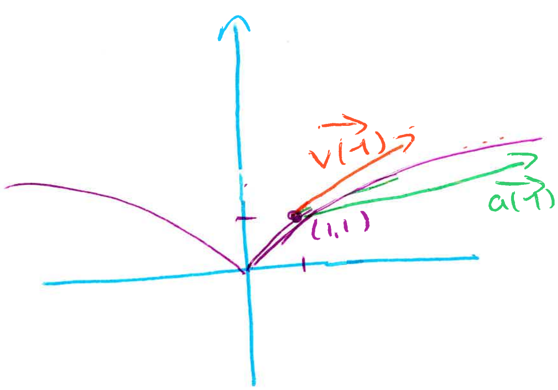
Motion: Velocity and Acceleration

We introduced space curves as the trajectory of a particle over the time. For a curve $\vec{r}(t)$:

- $\vec{r}(t)$ is the position vector at time t.
- $\vec{r}'(t)$ is the velocity vector at time t.
- $|\vec{r}'(t)|$ is the speed at time t
- $\vec{r}''(t)$ is the acceleration at time t (it is a vector)
- $\int_0^t \vec{r}'(t)\,dt$ is the displacement vector.

Example

The position vector of an object moving in a plane is given by $\vec{r}(t) = t^3\vec{i} + t^2\vec{j}$. Find its velocity, speed and acceleration when $t=1$, and illustrate it geometrically.



Fixing $x=t^3$ and $y=t^2$, we get $y = x^{2/3}$

Its velocity is
$$\vec{v}(t) = \vec{r}'(t) = \langle 3t^2, 2t\rangle,$$
its speed is
$$|\vec{v}(t)| = \sqrt{9t^4+4t^2},$$
and its acceleration is
$$\vec{a}(t) = \vec{v}'(t) = \langle 6t, 2\rangle.$$

When $t=1$:
$\vec{v}(1) = \langle 3,2\rangle, \quad \vec{a}(1) = \langle 6,2\rangle$

Example

A moving particle starts at an initial position $\vec{r}(0) = \langle 1,0,0 \rangle$ with initial velocity $\vec{v}(0) = \vec{i} - \vec{j} + \vec{k}$. Its acceleration is $\vec{a}(t) = 4t\vec{i} + 6t\vec{j} + \vec{k}$. Find its velocity and position at time t

We know that

$$\vec{v}(t) = \int_0^t \vec{a}(t)\,dt + \vec{v}(0).$$

Hence,

$$\vec{v}(t) = \int_0^t \langle 4t, 6t, 1 \rangle dt + \langle 1,-1,1 \rangle$$

$$= \langle 2t^2, 3t^2, t \rangle + \langle 1,-1,1 \rangle.$$

$$= \langle 2t^2+1, 3t^2-1, t+1 \rangle$$

is the velocity at time t.

In the same way,

$$\vec{r}(t) = \int_0^t \vec{v}(t)\,dt + \vec{r}(0),$$

and

$$\vec{r}(t) = \int_0^t \langle 2t^2+1, 3t^2-1, t+1 \rangle dt + \langle 1,0,0 \rangle.$$

$$= \langle \frac{2t^3}{3} + t,\ t^3 - t,\ \frac{t^2}{2} + t \rangle + \langle 1,0,0 \rangle$$

$$= \langle \frac{2t^3}{3} + t + 1,\ t^3 - t,\ \frac{t^2}{2} + t \rangle.$$

is the position vector at time t.

In general,

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^t \vec{a}(u)\,du$$

and

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^t \vec{v}(u)\,du.$$

Law of motion

We saw in the last unit Newton's second law of motion: $F = m\cdot a$. This of course has a vector version: $\vec{F} = m\vec{a}$.

Projectile motion

When a projectile is fired, it has a speed and the direction is important. Also, there is a force acting on it: the gravity, whose magnitude is given by the second law of motion $|\vec{F}| = |m\cdot\vec{a}| = m \cdot \underbrace{9.8\,m/s^2}_{\text{Gravitational acceleration}}$. The direction of $\vec{F}$ is the same as $-\vec{j}$.

Usually, we know what is the velocity and position at $t=0$.

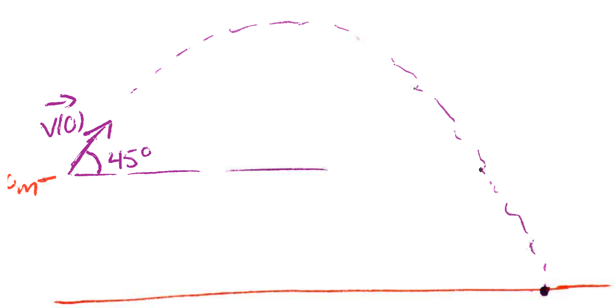
Example

A projectile is fired with muzzle speed 150 m/s and angle of elevation 45° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?

Initial position: $\vec{r}(0) = \langle 0, 10\rangle$

Initial velocity: $\vec{v}(0) = \langle 150\cdot\cos(45°), 150\sin(45°)\rangle$
$= \langle 75\sqrt{2}, 75\sqrt{2}\rangle$

Acceleration vector: $\vec{a}(0) = \langle 0, -9.8\rangle$



$$\vec{v}(t) = \vec{v}(0) + \int_0^t \langle 0, -9.8\rangle\, du$$
$$= \vec{v}(0) + \langle 0, -9.8t\rangle$$
$$= \langle 75\sqrt{2}, 75\sqrt{2} - 9.8t\rangle$$

Position vector: $\vec{r}(t) = \vec{r}(0) + \int_0^t \langle 75\sqrt{2}, 75\sqrt{2} - 9.8u\rangle\, du$

$$= \langle 0, 10\rangle + \langle 75\sqrt{2}t, 75\sqrt{2}t - 4.9t^2\rangle$$
$$= \langle 75\sqrt{2}t, 10 + 75\sqrt{2}t - 4.9t^2\rangle,$$

and the projectile touches the ground when the second component of $\vec{r}(t)$ is 0: this is at

$$t = \frac{-75\sqrt{2} \pm \sqrt{2\cdot 75^2 + 196}}{-9.8} \approx 21.74.$$

→ Only one answer. Otherwise $t<0$, and it makes no sense.

And the distance to the muzzle is $(21.74)\cdot 75\sqrt{2} \approx 2306$ meters

Components of acceleration

There are two parts to acceleration:

- The tangential acceleration is an acceleration in the direction of the motion (or of its tangent vector).
- The normal acceleration is an acceleration in the perpendicular direction. We should think of it as changing direction.

Definition

The unit normal vector of $\vec{r}(t)$ whose unit tangential vector is $\vec{T}(t)$ is given by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}.$$

It is called normal, because $\vec{N}(t)\cdot\vec{T}(t) = 0$. To see that, notice that $\vec{N}(t)\cdot\vec{T}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}\cdot\vec{T}(t) = \frac{1}{|\vec{T}'(t)|}(\vec{T}'(t)\cdot\vec{T}(t))$.

But we saw on Monday that if $|\vec{T}(t)|$ is constant, then $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$, and $\vec{T}(t)$ is a unit vector.

To get the components of acceleration, we differentiate $\vec{v}(t)$ in two different ways:

- We know that $\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$.
- Also, $\vec{v}(t) = \underbrace{|\vec{v}(t)|}_{\text{speed}}\cdot\vec{T}(t)$. Hence, $\vec{a}(t) = |\vec{v}(t)|'\cdot\vec{T}(t) + |\vec{v}(t)|\cdot\vec{T}'(t)$.

$$= |\vec{v}(t)|'\cdot\vec{T}(t) + |\vec{v}(t)|\,|\vec{T}'(t)|\cdot\vec{N}(t)$$

using $\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{T}'(t)|}{|\vec{v}(t)|}$

$$= |\vec{v}(t)|'\cdot\vec{T}(t) + |\vec{v}(t)|^2\cdot\kappa(t)\vec{N}(t).$$

Theorem

The tangential acceleration component is given by

$$a_T(t) = |\vec{v}(t)|',$$

and the normal acceleration component by

$$a_N(t) = K(t)\,|\vec{v}(t)|^2.$$

The acceleration is

$$\vec{a}(t) = a_T(t)\,\vec{T}(t) + a_N(t)\,\vec{N}(t)$$

Example

A particle moves with position function $\vec{r}(t) = \langle t^2, t^2, t^3 \rangle$. Find the components of acceleration.

Tangential acceleration:

$$\vec{v}(t) = \langle 2t, 2t, 3t^2 \rangle, \quad \text{and} \quad |\vec{v}(t)| = \sqrt{8t^2 + 9t^4}$$

Also, $$|\vec{v}(t)|' = \frac{16t + 36t^3}{2\sqrt{8t^2+9t^4}} = \frac{8t + 18t^3}{\sqrt{8t^2+9t^4}} = a_T(t).$$

Normal acceleration:

$$K(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|r'(t)|^3}$$

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} \right| = |\langle 6t^2, -6t^2, 0 \rangle| = 6\sqrt{2}\,t^2,$$

so

$$a_N(t) = \frac{6\sqrt{2}\,t^2}{|r'(t)|} = \frac{6\sqrt{2}\,t^2}{\sqrt{8t^2+9t^4}}$$

Reference: James STEWART. Calculus, 8th edition. §13.4.