Math 8 - Lecture 19

Functions of Several variables: level curves

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02/17/2020

In the past weeks, we looked at surfaces and defined their equations, that have two independant variables.

We now look at them as functions.

Definition

A function f of two variables is a rule that assigns to each ordered pair of real numbers (x,y) a unique real number $f(x,y)$, for (x,y) in a set D (the domain). The set of values $\{f(x,y) \mid (x,y) \in D\}$ is the range.

Example

The function $z = f(x,y) = 3x - 2y$ is the function that represents the plane $3x - 2y - z = 0$, which is the plane with normal vector $\langle 3, -2, -1 \rangle$ passing through the origin.

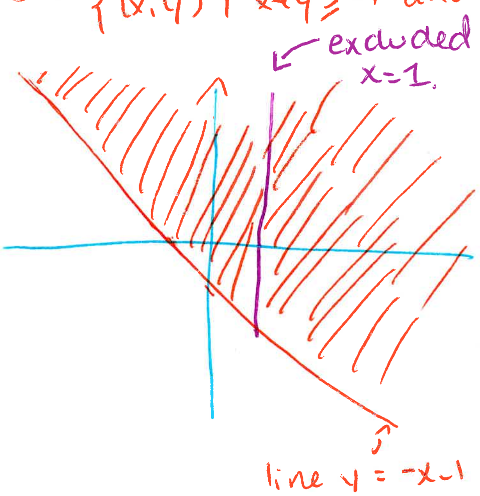
Its domain is $(\mathbb{R}, \mathbb{R})$ and its range is $\mathbb{R}$.

Example

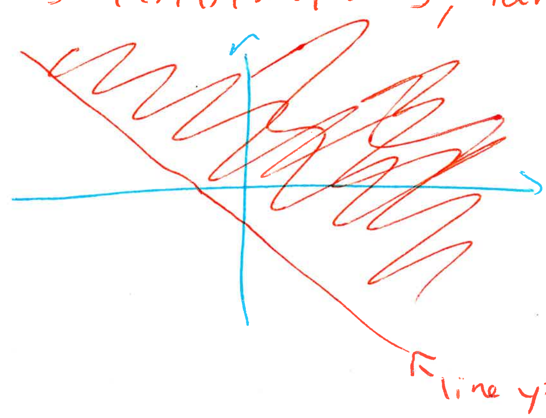
Find the domain and range of the following functions, and sketch the domain.

a) $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$, b) $f(x,y) = \sqrt{x+y+1}$, c) $f(x,y) = x \ln(y^2 - x)$.

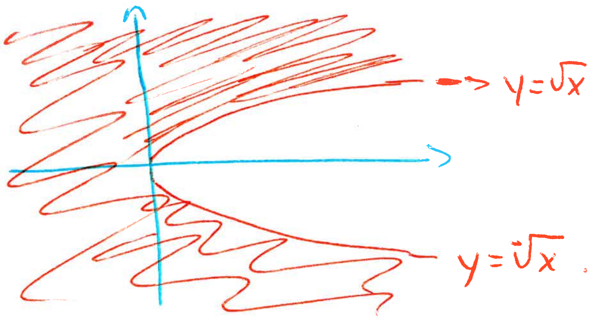
a) $D = \{(x,y) \mid x+y \geq -1 \text{ and } x \neq 1\}$, and the range is $\mathbb{R}$



b) $D = \{(x,y) \mid x+y \geq 1\}$, range: $z \geq 0$



c) $D = \{(x,y) \mid y^2 - x \geq 0\}$
$= \{(x,y) \mid y^2 \geq x\}$
$= \{(x,y) \mid y \geq \sqrt{x} \text{ or } y \leq -\sqrt{x}\}$



The range is $\mathbb{R}$.
To see it, look at all the values of $f(x,y)$ when $x=1$. The range of $f(y) = \ln(y^2-1)$ is $\mathbb{R}$.

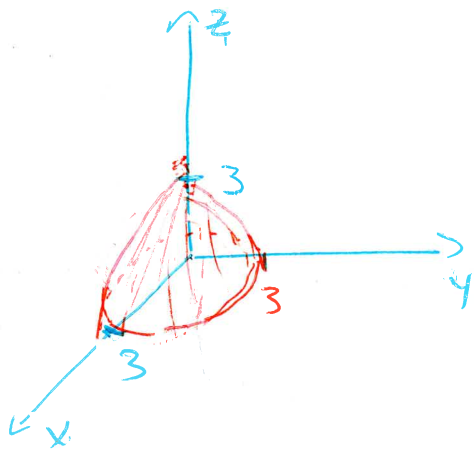
Example

Find the domain and range of $g(x,y) = \sqrt{9-x^2-y^2}$.
Can you sketch the graph of $z = \sqrt{9-x^2-y^2}$.

$D = \{(x,y) \mid x^2+y^2 \leq 9\}$, which is the disk of radius 3 centered at the origin.

The range is $0 \leq z \leq 3$.

You can rewrite $z = \sqrt{9-x^2-y^2}$ as $z^2+x^2+y^2 = 9$, with the constraint that $z \geq 0$. Hence, it is a half-sphere of radius 3 centered at origin that lies above the xy-plane.

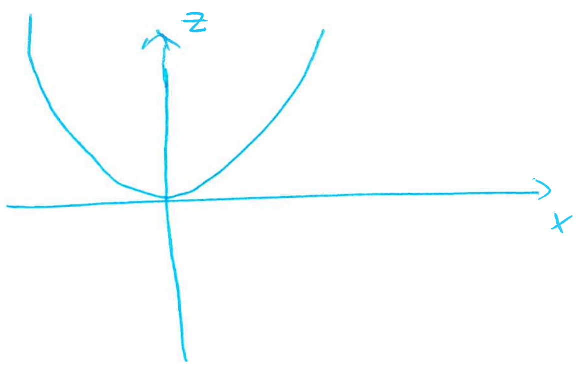


Problem

Sketch the function $f(x,y) = x^2 - 3y^2$.

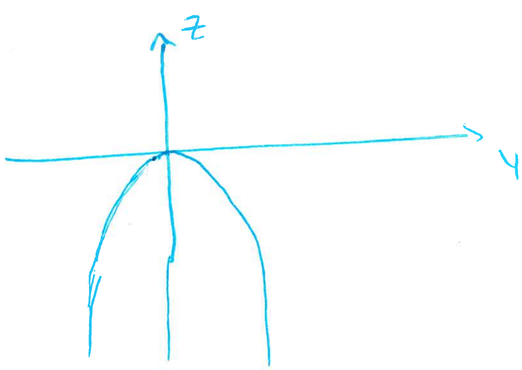
Solution

The intersection with the plane $y=0$ is



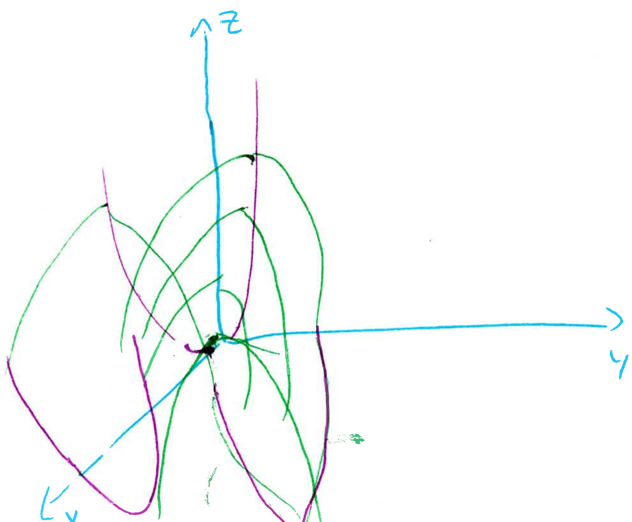
The intersection with the plane $y=c$ always looks the same, except that it crosses the z-axis lower when $y \neq 0$.

The intersection with $x=0$ is



The intersection with $x=c$ looks the same, but the top of the curve will be above the xy-plane.

The curve looks like



It is a horse saddle (or a pringle).

If you really want to see the 3D-features, draw it on Geogebra

Level curves

The level curves of a function f of two variables are the curves with equation $f(x,y) = k$, where k is a constant in the range of f.

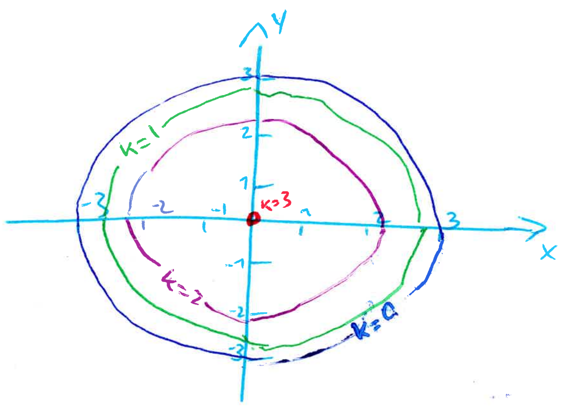
We project them to the xy-plane.

Example

The level curves on a tepographical map.

Example

Sketch the curves for $k \in \{0,1,2,3\}$ for $f(x,y) = \sqrt{9-x^2-y^2}$



$k=0 \quad 9 = x^2+y^2$

$k=1 \quad 8 = x^2+y^2$

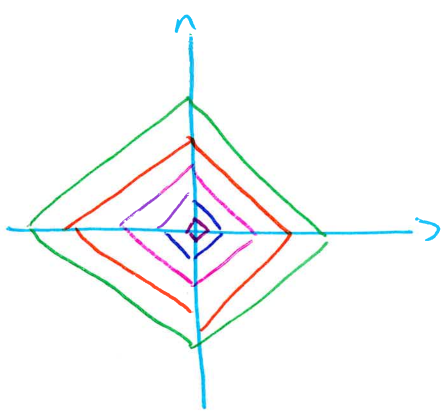
$k=2 \quad 5 = x^2+y^2$

$k=3 \quad 3 = \sqrt{9-x^2-y^2}$

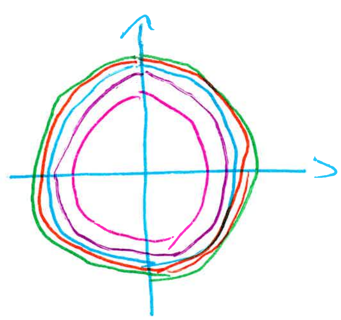
$\Rightarrow x=y=0.$

Example

Describe the functions that can be associated with the level curves.

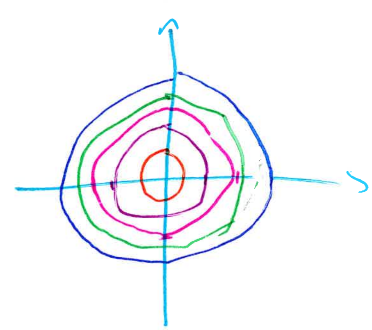


(a pyramid)



(a paraboloid, $x^2+y^2=z$)

The level curves are similar to a sphere.

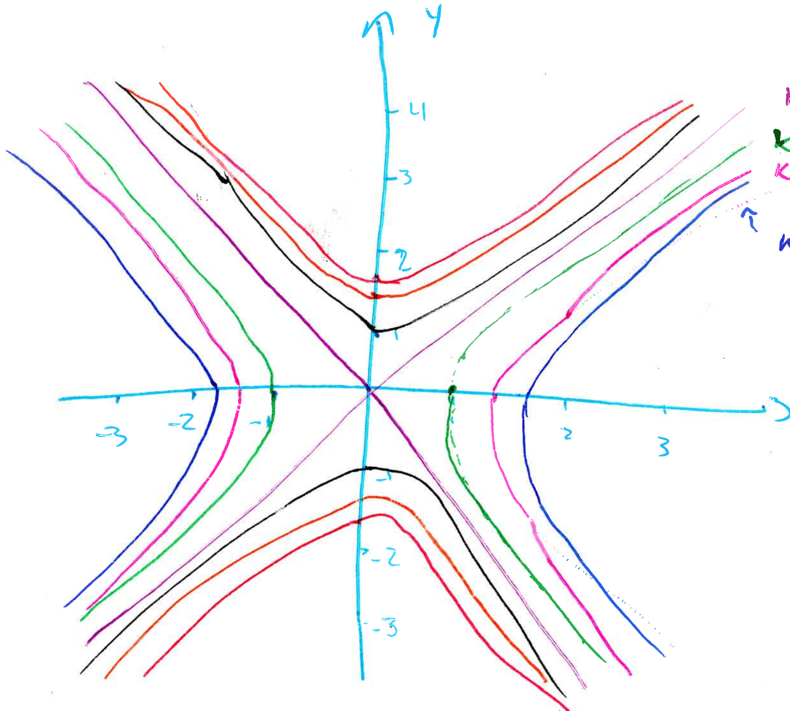


(a cone $x^2+y^2=z^2$)

Note that some functions have disconnected components for their level curves.

Example

Draw the level curves of $x^2 - y^2 = f(x,y)$ for $k \in \{-3,-2,-1,0,1,2,3\}$.



$k=0$ $(x^2 = y^2 \Rightarrow y = \pm x)$.

$k=1$ $y = \pm\sqrt{x^2-1}$

$k=2$ $y = \pm\sqrt{x^2-2}$

$k=3$ $y = \pm\sqrt{x^2-3}$

$k=-1$ $y = \pm\sqrt{x^2+1}$

$k=-2$ $y = \pm\sqrt{x^2+2}$

$k=-3$ $y = \pm\sqrt{x^2+3}$

Software demo: Geogebra.

Reference: James STEWART. Calculus, 8th edition. §14.1