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Math 8 - Lecture 1

- Ice breaker, go-around-the-table: 1st name, expected major (if any), what place feels home to you?

10 mins

- Syllabus: some highlights

10 min

  - assignments
  - Honor Principle, how it applies
  - Tutorials office hours, x-hour
  - Policies
  - Question?

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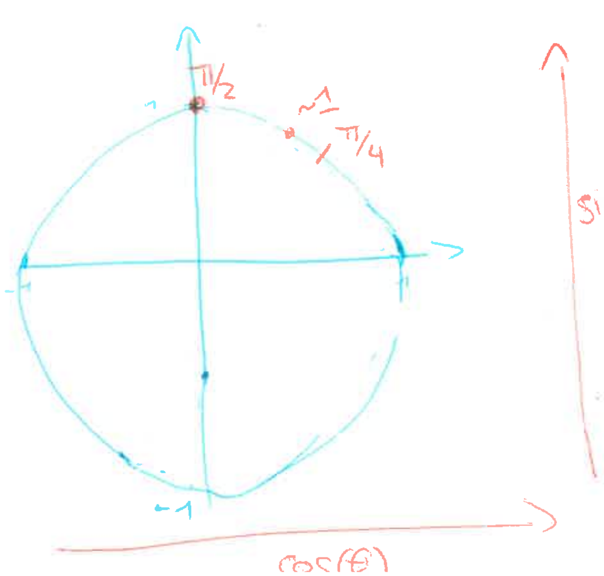
One more policy: no more calculator.

- A first exercise in that setting: compute $\sin(1)$.
  - What does $\sin(1)$ mean?
  - What is 1? Angle, radians
- What we know about it:
  - $-1 \leq \sin(x) \leq 1$ for $\underbrace{x \in \mathbb{R}}_{\text{notation: } x \text{ a real number.}}$
  - $\frac{\pi}{4} \leq 1 \leq \frac{\pi}{2}$ and, with the picture, we know that

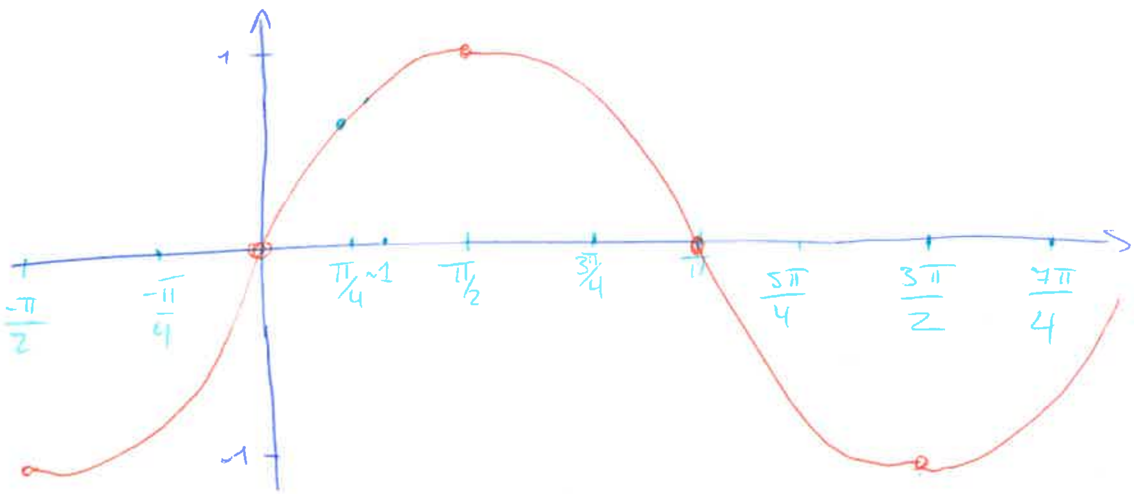
$$\sin\left(\frac{\pi}{4}\right) \leq \sin(1) \leq \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{\sqrt{2}}{2} \leq \sin(1) \leq 1.$$

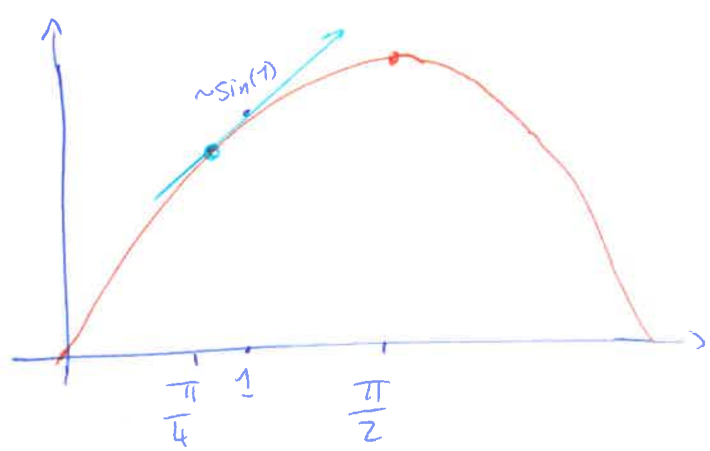


(2)

The other thing we have an idea of is the graph of sin(x):



To get an approximation of sin(1), we can start from $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, and draw the tangent line close to it



Then, a closer approximation of sin(1) would be

$$\sin(1) \approx \sin\left(\frac{\pi}{4}\right) + \sin'\left(\frac{\pi}{4}\right) \cdot \left(1-\frac{\pi}{4}\right)$$

$$= \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \cdot \left(1-\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2}\left(1+1-\frac{\pi}{4}\right) \approx 0.707 \cdot 1.21 \approx 0.8591$$

(Exact answer is close to 0.8415)

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On the graph, we see that the point on the straight line (the one we computed) seems to be above the actual value of $\sin(1)$.

This means, we should "bend down" our approximation of $\sin(x)$ to get a better estimation of $\sin(1)$.

1-2 all Question: How can we do it?

1-1

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Recap An outline of the method:

We need two things:

- a nice (i.e. smooth) function f that is hard to compute.
- a real number (in the domain of f), around which we want to get an estimation of f

We will approximate f by a function g that is easy to compute (i.e. a polynomial)

For g to be appropriate, one must have,

$$f(a) = g(a),$$

and, ideally,

$$f'(a) = g'(a),$$
$$f''(a) = g''(a),$$
$$\vdots \qquad \vdots$$
$$f^{(n)}(a) = g^{(n)}(a).$$

We then say that g approximates f in the neighborhood of a.

Definition

The n-th degree Taylor polynomial of f centered at a, denoted $T_n(x)$, is the unique polynomial of degree n such that

$$f(a) = T_n(a)$$
$$f'(a) = T_n'(a)$$
$$\vdots \qquad \vdots$$
$$f^{(n)}(a) = T_n^{(n)}(a).$$

Question. How do we find what T is like?

4x4

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Reminder: a polynomial of degree n is a function of the form

$$p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n.$$

If they are lost after 3 minutes:

find the problem with

$$T_2(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2.$$

Definition (continued)

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f^{(3)}(a)(x-a)^3}{6} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

A Maclaurin polynomial is a Taylor polynomial centered at $a=0$.

2x2

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Exercise: Verify that $T_n(x)$ and $f(x)$ have the same n-th derivative.

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Solution

This page will skip if we don't have time. 5

The terms of degree smaller than n will be canceled by the n-th derivative. (If that sounds hard, come to the x-bar on Tuesday).

So we only need to verify that the n-th derivatives of f and $\dfrac{f^{(n)}(a)(x-a)^n}{n!}$ are equal when evaluated in a.

Moreover,

$$\left(\frac{f^{(n)}(a)\,(x-a)^n}{n!}\right)' = \frac{f^{(n)}(a)}{n!}\left((x-a)^n\right)'$$

Why? The variable is x.

Finally,

$$\left((x-a)^n\right)^{(n)} = \left(n\,(x-a)^{n-1}\right)^{(n-1)}$$

$$= \left(n\cdot(n-1)\cdot(x-a)^{n-2}\right)^{(n-2)}$$

$$= \ldots$$

$$= \left(n\cdot(n-1)(n-2)\cdots(n-i)\cdot(x-a)^{n-i-1}\right)^{(n-i-1)}$$

$$= \ldots$$

$$= \Big(\underbrace{n\cdot(n-1)(n-2)\cdots(n-n+1)}_{n!}\cdot\cancel{(x-a)^{n-n}}\Big)^{\cancel{(n-n)}}$$

$$= n!$$

Thus, $\left(\dfrac{f^{(n)}(a)\,(x-a)^n}{n!}\right)^{(n)} = \dfrac{f^{(n)}(a)}{n!}\,n! = f^{(n)}(a)$, the n-th derivative of f.

Example

Approximate $e^{0.05}$ using a third-degree Taylor polynomial.

Observation: We know how to compute e^0 and the derivatives of e^x evaluated in $x=0$.

Let $f(x)=e^x$.

$$f(x)=e^x \qquad f(0)=1$$

$$f'(x)=e^x \qquad f'(0)=1$$

$$f''(x)=e^x \qquad f''(0)=1$$

$$f^{(3)}(x)=e^x \qquad f^{(3)}(0)=1.$$

So $$T_3(x) = 1 + 1\cdot\frac{(x-0)}{1} + 1\cdot\frac{(x-0)^2}{2} + \frac{1\,(x-0)^3}{6}.$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Moreover,

$$e^{0.05} \approx T_3(0.05) = 1 + 0.05 + \frac{(0.05)^2}{2} + \frac{(0.05)^3}{6}$$

$$= 1 + 0.05 + 0.00125 + \frac{0.000125}{6}$$

$$\approx 1.051$$

So $e^{0.05}$ is estimated by 1.051.

Example

Compute the 9-th order Maclaurin polynomial of $f(x) = \sin(x)$.

- Computations of derivatives:

$$f(x) = \sin(x) \qquad f(0) = 0$$
$$f'(x) = \cos(x) \qquad f'(0) = 1$$
$$f''(x) = -\sin(x) \qquad f''(0) = 0$$
$$f^{(3)}(x) = -\cos(x) \qquad f^{(3)}(0) = -1$$
$$f^{(4)}(x) = \sin(x) \qquad f^{(4)}(0) = 0$$
$$\vdots \qquad \vdots$$
$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

So,

$$T_0(x) = f(0) = 0$$
$$T_1(x) = f(0) + f'(0)(x-0) = x$$
$$T_2(x) = T_1(x) + \frac{f''(0)\, x^2}{2!} = x$$
$$T_3(x) = T_2(x) + \frac{f^{(3)}(0)\, x^3}{3!} = x - \frac{x^3}{3!}$$
$$\vdots$$
$$\boxed{T_9(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}}$$

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Final question:

What about sin(1) from the beginning?

We approximated sin(x) around $a = \pi/4$.

$$(\sin(x))' = \cos(x), \quad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$(\sin(x))'' = -\sin(x), \quad -\sin\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$$

$$(\sin(x))''' = -\cos(x), \quad -\cos\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$$

With Geogebra, plot

$$f = \sin(x)$$

$$T_0(x) = \frac{\sqrt{2}}{2}$$

$$T_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$

$$T_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$$

$$T_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$$

and

$$f(a) \approx 0.841$$

$$T_0(a) \approx 0.707$$

$$T_1(a) \approx 0.859$$

$$T_2(a) \approx 0.843$$

$$T_3(a) \approx 0.841$$