Math 8 - Lecture 20 Nadia Lafrenière

Limits and continuity for functions of several variables 2/19/2020

Problem

What is the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \ ?$$

If that limit exists, then it should be equal to

$$\lim_{y\to 0} \frac{-y^2}{y^2} \quad \text{and to} \quad \lim_{x\to 0} \frac{x^2}{x^2}$$

But these are respectively -1 and 1. So the limit is not defined
(picture on Geogebra)

Definition

Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then, we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L,

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if, regardless of the way we approach (a,b), the limit exists and always has the same value.

Notice that this definition also holds for functions of one variable. However, in the case of functions of one variable, there are only two ways of approaching it, whereas there are infinitely many ways for functions of several variables.

Example

If $f(x,y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist ?

Two ways of approaching $(0,0)$ are on the lines $x=y$ and $x=0$

(i) If $x=y$,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

(ii) If $x=0$,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0\,y}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

Since the two paths do not lead to the same answer, the limit does not exist.

Example

Does $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$ exist?

Choosing to approach it from $x=0$, we get

$$\lim_{y\to 0} \frac{0\cdot y^4}{0^2+y^8} = \lim_{y\to 0} \frac{0}{y^8} = 0.$$

Choosing $y=0$ or $x=y$ would lead to the same.

However, choosing $x=y^4$, we get

$$\lim_{y\to 0} \frac{y^8}{y^8+y^8} = \frac{1}{2}.$$

The limit does not exist.

Polar coordinates

On top of the (x,y)- way of writing every point in the plane, there is another way of describing uniquely every point: by polar coordinates.

Proposition

Let $P=(x,y)$ be a point in the plane, and let $\vec{v}$ be its position vector (i.e. $\vec{v}=\langle x,y\rangle$). Then, P can be uniquely identified by the following data:

- r, the magnitude of $\vec{v}$. That is $r=\sqrt{x^2+y^2}$.
- θ, the angle between the x-axis and $\vec{v}$, being counterclockise:

  Since $\tan(\theta)=\dfrac{\sin(\theta)}{\cos(\theta)}=\dfrac{y/r}{x/r}=\dfrac{y}{x}$, $\theta=\arctan\left(\dfrac{y}{x}\right)$.

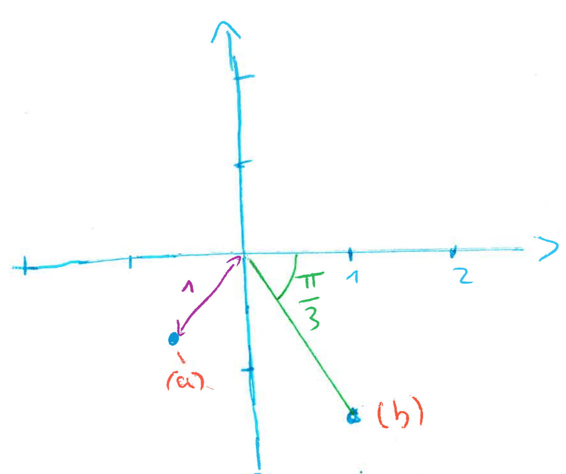
Also, knowing r and θ, one can go back to (x,y):

$x=r\cos(\theta)$, $y=r\sin(\theta)$.

Example

Plot the points given in polar coordinates (r,θ):

(a) $(1, 5\pi/4)$ (b) $(2, -\pi/3)$



Example

Does $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ exist?

Choosing $y=0$,

$$\lim_{x\to 0} \frac{3x^2 0}{x^2+0^2} = \lim_{x\to 0} \frac{3\,0}{x^2} = 0$$

Choosing $x=0$, $x=y$ or $x^2=y$, we get 0 as the limit.

Try to prove this is true:

We change the equation to polar coordinates. We know $(x,y)\to(0,0)$ when $r\to 0$. Then, we have

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{3r^3\sin(\theta)\cos^2(\theta)}{r^2} = \lim_{r\to 0} 3r\sin(\theta)\cos^2(\theta)$$
$$= 0.$$

Hence, the limit exists, since, regardless of θ, it goes to 0.

Continuity

A function f of two variables is called <u>continuous</u> at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say f is continuous on D if f is continuous at every point (a,b) in D.

Example

- A polynomial is always continuous: $x + x^2y$ is continuous everywhere on $\mathbb{R}^2$.

- $$g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$ is not continuous in $(0,0)$, since $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist.

- $$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$ is continuous in $(0,0)$

- Where is $\arctan(x/y)$ continuous?
  - $\arctan(x)$ is continuous everywhere
  - x/y is not defined in $y=0$, and continuous everywhere else.
  - So $\arctan(x/y)$ is continuous everywhere, except where $y=0$

Extra problems

(i) Prove that $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ exists.

(ii) Compute $\lim_{(x,y)\to(3,2)} e^{\sqrt{2x-y}}$. Where is $e^{\sqrt{2x-y}}$ continuous?

(iii) Does $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$ exist?

Reference: James STEWART. Calculus, 8th edition §14.2 and §10.3.