Math 8 - Lecture 21
Partial derivatives

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We continue our exploration of functions of several variables, by looking at their derivatives. Just like we did for defining limits and continuity, we want to restrict the functions to functions of one variable.

Definition

The **partial derivative** of $f(x,y)$ with respect to x at (a,b), denoted $f_x(a,b)$ is $f_x(a,b) = g'(a)$, where $g(x) = f(x, \underset{\text{fixed.}}{b})$.

Similarly, $f_y(a,b) = h'(b)$, where $h(y) = f(a,y)$.

Since g and h are defined as function of one variable, we can define their derivative in the usual way (with differentiation's rules, or with the definition with the limit.

Notation

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_x$$

and $$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = f_y$$

Rule for finding partial derivatives of $f(x,y)$

(i) To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

(ii) To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Example

If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Solution

$$f_x(2,1) = (3x^2 + 2xy^3)_{x=2,y=1} = 12 + 4 = 16.$$

$$f_y(2,1) = (3x^2y^2 - 4y^2)_{x=2,y=1} = 12 - 4 = 8.$$

Example

If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(i) $\frac{\partial f}{\partial x} = \frac{1}{1+y} \cos\left(\frac{x}{1+y}\right)$ with the chain rule

(ii) $\frac{\partial f}{\partial y} = \frac{-x}{(1+y)^2} \cos\left(\frac{x}{1+y}\right)$

Implicit definition

In one dimension, a function $y = f(x)$ is defined implicitly if it is not defined by an equation with y isolated (on one side of the equation).

Example

The relation $x^2 + y^2 = 25$ is the implicit definition of the two functions $y = \pm\sqrt{25 - x^2}$

Implicit functions can be differentiated using the chain rule:

$$\frac{\partial}{\partial x}(x^2 + y^2) = \frac{\partial}{\partial x} 25$$

$$\Rightarrow \quad 2x + 2y \frac{\partial y}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial y}{\partial x} = \frac{-2x}{2y} = -\frac{x}{y}.$$

The same thing can be done for functions of two variables.

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $e^z = xyz$.

(i) $$\frac{\partial}{\partial x} e^z = \frac{\partial}{\partial x}(xyz) = y\,\frac{\partial}{\partial x}(xz) \overset{\text{product rule.}}{=} y\,(x z' + z)$$

On the other hand,

$$\frac{\partial}{\partial x} e^z = e^z \cdot \frac{\partial z}{\partial x} = e^z z'$$

Here, remember z is a function of x.

Thus, solving for z', we get

$$\frac{\partial z}{\partial x} = z' = \frac{yz}{e^z - xy}.$$

(ii) $$\frac{\partial}{\partial y} e^z = e^z \frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial y} xyz = x\,\frac{\partial}{\partial y}(yz) = x\left(z + y\frac{\partial z}{\partial y}\right)$$

Hence,

$$\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

Notice that, by the symmetry of x and y in the equations (i.e. they play the same role), we could have deduced this formula by exchanging x and y.

Higher derivatives (second-order, third derivative, etc.)

If f is a function of two variables, then so are f_x and f_y, so we can define second partial derivatives: $f_{xy}, f_{xx}, f_{yx}, f_{yy}$.

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \qquad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

⚠ ∂^2 does not mean the square of the derivative.

Example

Find all the second partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$.

On page 2, we already found that

$$f_x(x,y) = 3x^2 + 2xy^3 \quad \text{and} \quad f_y(x,y) = 3x^2y^2 - 4y$$

There fore,

$$f_{xx}(x,y) = 6x + 2y^3 \qquad f_{yx}(x,y) = 6xy^2$$

$$f_{xy}(x,y) = 6xy^2 \qquad f_{yy}(x,y) = 6x^2y - 4.$$

Notice that $f_{xy} = f_{yx}$. This is not a coincidence...

Theorem (Clairaut, Yang, Euler, Schwartz...)

Let f be a function of x and y.

If f_{xy} and f_{yx} are continuous near (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Example

Let $f(x,y) = \ln(x+2y)$.

$$f_{xy} = \frac{\partial}{\partial x} \frac{2}{x+2y} = \frac{-2}{(x+2y)^2} \quad \text{and} \quad f_{yx} = \frac{\partial}{\partial y} \frac{1}{x+2y} = \frac{-2}{(x+2y)^2},$$

and indeed $f_{xy} = f_{yx}$.

Example

If $f(x,y) = x^3y^2 + \arcsin(x)$, find f_{xy}.

Using $f_{yx} = f_{xy}$, we get

$$f_{yx} = \frac{\partial}{\partial x} 2x^3y = 6x^2y = f_{xy}$$

Reference: James STEWART. Calculus, 8th edition. § 14.3.