Math 8 - Lecture 23 Nadia Lafrenière

Tangent planes. 02/24/2020

A very important idea of calculus is that one can approximate locally a function of one variable with a straight line.

Can we do that more generally for functions of several variables?

Today, we do it for functions of two variables with tangent planes.

The main idea: use the partial derivatives as "slopes" for our plane.

Theorem

If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then

there is a plane that is tangent to f in (a,b). We will make it more formal tomorrow.

Proposition

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z=f(x,y)$ at the point $(a,b,f(a,b))$ is

$$z = \underline{f(a,b)} + \underline{f_x(a,b)}\,(x-\underline{a}) + \underline{f_y(a,b)}\,(y-\underline{b})$$

$\underline{\ \ }$ = constant terms

Remarks

- This is the equation of a plane.
  Note that it is not possible to have exponents for x, y and z.
- The point $(a,b,f(a,b))$ is on that plane.

Example

We will prove geometrically tomorrow that $f(x,y)=x^2+y^2$ has tangent plane $z=2x+4y-5$, in $(1,2,5)$

The partial derivatives of f are $f_x=2x$ and $f_y=2y$.

Hence, $f_x(1,2)=2$ and $f_y(1,2)=4$.

they are continuous, so the plane is

$$z=5+2(x-1)+4(y-2)$$
$$=5+2x-2+4y-8$$
$$=2x+4y-5$$

Linear approximations

A very important goal here is to approximate a differentiable function f near (a,b)

$$f(x,y)\approx f(a,b)+f'(a,b)\cdot\langle x-a,y-b\rangle$$

directional derivative $\langle f_x(a,b), f_y(a,b)\rangle$.

$$=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

(Intuition)

That approximation is a linear approximation or the tangent plane approximation.

How it works:

- $\langle x-a,y-b\rangle$ is a vector in the xy-plane.
- We want to project that displacement onto the tangent plane to $f(x,y)$. In the x-direction, the "slope" of the tangent plane is $f_x(a,b)$, which means that if $x-a=1$, the displacement is $f_x(a,b)$. If it is smaller or bigger, the difference on the plane will be smaller or bigger.
- The same thing holds in the y-direction.

Example

Show that $f(x,y) = xe^{xy}$ has a tangent plane at $(1,0)$ and approximate $f(1.1,-0.1)$

We have. $f_x(x,y) = e^{xy} + xye^{xy}$, that is continuous

$f_x(1,0) = 1$.

and $f_y(x,y) = x^2 e^{xy}$ (also continuous) and

$f_y(1,0) = 1$.

So, near $(1,0)$,

$$f(x,y) \approx f(1,0) + 1\cdot(x-1) + 1\,(y-0)$$
$$= 1 + (x-1) + y$$
$$= x+y$$

In particular,

$$f(1.1,-0.1) \approx 1$$

(The actual value of $f(1.1,-0.1) \approx 0.985$).

Differentials

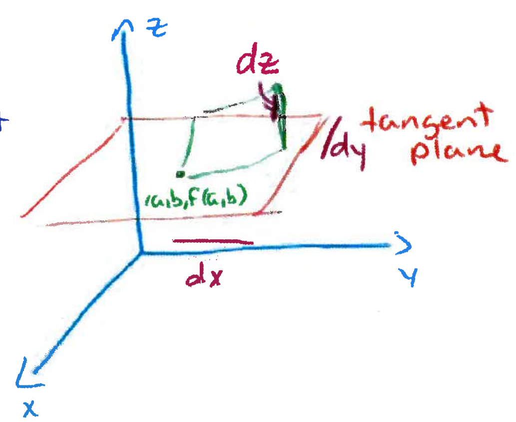
The differentials dx and dy represent small movements in the direction of the x- and y- axis respectively.

For functions of two variables, dx and dy are independent

The total differential, dz, is defined by

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

If the plane is the tangent plane at $(a,b,f(a,b))$, then



$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

whenever (x,y) is close to (a,b).

Example

- If $z = f(x,y) = x^2 + 3xy - y^2$, find dz.
- Compare it with Δz when x changes from 2 to 2.05 and y from 3 to 2.96 (Note that $f(2.05, 2.96) = 13.6449$)

Here, $\Delta z = f(2.05, 2.96) - f(2,3)$

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$
$$= (2x+3y)\,dx + (3x-2y)\,dy$$

- When x changes from 2 to 2.05, $dx = 0.05$

when y changes from 3 to 2.96, $dy = -0.04$

Then, $dz = \underbrace{(2(2)+3(3))}_{\text{evaluated at } (a,b)=(2,3)} \cdot 0.05 + (3\cdot 2 - \cancel{2}\,2\cdot 3)(-0.04)$

$= 0.65$

Also, $\Delta z = 13.6449 - f(2,3)$
$= 13.6449 - (4+18-9)$
$= 13.6449 - 13$
$= 0.6449$

Notice that $\Delta z \approx dz$, but dz is much easier to compute.

Example

Find a linear approximation at $(0,0)$ of $e^x \cos(xy) = f(x,y)$
Also, find the differential and difference with $f(0.1, -0.1)$, provided that $f(0.1, -0.1) \approx 1.1051$

$\hookrightarrow \Delta z = f(0.1, -0.1) - f(0,0)$.

The linear approximation is

$$f(x,y) \approx e^0 \cos(0) + 1\cdot(x-0) = 1+x.$$

because

$f_x(x,y) = e^x\cos(xy) - e^x y \sin(xy)$ $\rightarrow$ at $(0,0)$, this is 1. and $f_y(x,y) = -xe^x \sin(xy)$. $\rightarrow$ at $(x,y)=(0,0)$, this is 0.

The differential is then $dz = dx$, so when x moves from 0 to 0.1, this is $dz = 0.1$.

Also, $\Delta z = 1.1051 - 1 = 0.1051$ and $dz \approx \Delta z$.

Reference: James STEWART. Calculus, 8th edition. §14.4.