Math 8 - Lecture 23 Nadia Lafrenière

Derivatives as Linear Approximations 2/25/2020

What is the meaning of the derivative, geometrically?

- For real-valued functions of one variable: it is the slope of the function. It also gives the tangent line that approximates the function close to a point.
- For vector-valued functions: it is the tangent vector to that function, and it gives the direction of the line tangent to the curve.
- For surfaces? There are many slopes...
  - In some cases, it gives a tangent plane (if that exists).
  - We can use it to approximate the value of a function.

Example

Let $f(x,y) = x^2+y^2$ be a paraboloid.

- Does f admit a tangent plane in $(1,2,5)$?

We will see later that it is the case, since its equation is a polynomial.

- What is the slope of f in $(1,2,5)$?
  - The slope of $g(x) = x^2+4$ is $2x$, which in $x=1$ is worth 2.

    So the line $\{z = 2x+3, y=2\}$ is tangent to the paraboloid at $(1,2,5)$.
  - Equivalently, with $h(y) = y^2+1$ in $y=2$, the line $\{z=4y-3, x=1\}$ is tangent to f in $(1,2,5)$.

Definition

Let f be a function of two variables and P be a plane that is not vertical, so that $f(a,b) = P(a,b) = c$, for some $a,b,c \in \mathbb{R}$.

We say that P is tangent of f in (a,b) if

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$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-P(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0.$$

Meaning: The numerator is the distance between f and P (in the z-coordinate) and the denominator is the distance between (x,y) and (a,b) (where f and P are equal).
As we get closer to (a,b) (in any direction), we want that the distance between f and P be very small.

Example

Check that the plane containing the lines $\{2x+3=z,\ y=2\}$ and $\{z=4y-3,\ x=1\}$ is tangent to $f(x,y)=x^2+y^2$ in $(1,2,5)$.

The first line can be written as $\langle t, 2, 2t+3\rangle = \vec{r_1}(t)$ and the second one as $\langle 1, t, 4t-3\rangle = \vec{r_2}(t)$. To find the plane containing both, we use the cross product:

$$\langle 1,0,2\rangle \times \langle 0,1,4\rangle = \langle -2,-4,1\rangle,$$

so the plane has equation $-2x-4y+z=-5$, or $z=2x+4y-5$.
It is tangent to f if the following limit is 0:

$$\lim_{(x,y)\to(1,2)} \frac{x^2+y^2-(2x+4y-5)}{\sqrt{(x-1)^2+(y-2)^2}} = \lim_{(x,y)\to(1,2)} \frac{(x^2-2x+1)+(y^2-4y+4)}{\sqrt{(x-1)^2+(y-2)^2}}$$

$$= \lim_{(x,y)\to(1,2)} \sqrt{(x-1)^2+(y-2)^2}$$

$$= 0.$$

Then, this is the plane tangent to f.

See on Geogebra.

Proposition

If there exists a tangent plane to f, it contains the lines tangent to f on a given plane

Example

The function $f(x,y) = \frac{2xy}{\sqrt{x^2+y^2}}$ does not admit a tangent plane at $(0,0)$.*

- Setting $x=0$, the function: $g(y) = \frac{0}{\sqrt{y^2}} = 0$ leads to $(0,y,0)$ being in the values of the function
- Setting $y=0$, the function $h(x) = 0$ gives $(x,0,0)$ being in the function. (x-axis)
- Since $x=0$ and $y=0$ are part of the surface, the tangent plane, if it exists, should be $z=0$.
- However, on the line $x=y$, $f(x,y) = \frac{2x^2}{\sqrt{2x^2}} = \sqrt{2}\,|x|$, and we cannot define a line tangent to the function $\sqrt{2}|x|$ close to 0.

So there is no tangent plane. Geogebra.

* Notice that the partial derivatives are not continuous...

Definition

The function $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if its graph has a non-vertical tangent plane at the point $(a,b,f(a,b))$.

If the plane is $P(x,y) = sx + ty + d = z$, then the <u>derivative</u> of f at (a,b) is $f'(a,b) = \langle s,t \rangle$.

It is also called the <u>directional derivative</u>.

Application

Given a differentiable function f in (a,b), we can approximate the values of f whenever (x,y) is close enough to (a,b):

$$f(x,y) \approx f(a,b) + f'(a,b) \underset{\substack{\uparrow \\ \text{dot} \\ \text{product}}}{\cdot} \langle x-a,\ y-b \rangle .$$

Example

Let $f(x,y) = x^2 + y^2$. (we computed the tangent plane on p. 2)

Then, $f(1.1, 1.9) \approx 5 + 2\cdot(0.1) + 4(-0.1) \approx 4.8$.

by taking $a = 1$, $b = 2$

Checking with a calculator, we get $1.1^2 + 1.9^2 = 4.82$

References:
- Marcia's handouts (see Section 2)
- James STEWART. Calculus, 8^{th} edition, §14.4