Math 8 - Lecture 24

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Chain Rule

Theorem (Chain rule, version 1)

Suppose that $z=f(x,y)$ is a differentiable function and x and y are themselves functions of t: $x=g(t)$ and $y=h(t)$, and they are both differentiable.

Then, z is differentiable in t, and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

partial derivative "∂"

derivative of a function of one variable "d"

Example

If $z = x^2y+3xy^4$, where $x=\sin 2t$ and $y=\cos t$, find $\frac{dz}{dt}$ when $t=0$.

With the chain rule, we find

$$\frac{\partial z}{\partial x} = 2xy+3y^4 \quad \text{and} \quad \frac{\partial z}{\partial y} = x^2+12xy^3.$$

Also,

$$\frac{dx}{dt} = 2\cos(2t) \quad \text{and} \quad \frac{dy}{dt} = -\sin t.$$

Hence,

$$\frac{dz}{dt} = 2(2xy+3y^4)\cos(2t) - (x^2+12xy^3)\sin t$$

For example, at $t=0$, $x=0$ and $y=1$, which gives

$$\frac{dz}{dt} = 6\cos(0) - 0\sin(0) = 6$$

Notice that

(i) We can leave x and y in the equation, as they were part of the question.

(ii) It is much easier to use the chain rule than to substitute first and then compute the derivative with the product rule.

Example

The pressure (in kPa), the temperature (in K) and the volume (in L) of a mole of an ideal gas are related through the equation

$$P = 8.31 \frac{T}{V}.$$

Find the rate at which the pressure is changing (in kPa/s) when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is increasing at a rate of 0.2 L/s, starting at 100 L.

Using the chain rule,

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt}$$

$$= \frac{8.31}{V} 0.1 + \frac{-8.31 T}{V^2} \cdot 0.2$$

$$= \frac{8.31}{100}\left(0.1 - \frac{300}{100} \cdot 0.2\right)$$

$$= \frac{8.31}{100} \; -0.5$$

$$= -0.04155$$

So the pressure is decreasing at a rate of -0.04155 kPa/s.

Theorem (Chain rule, version 2)

Suppose that $z = f(x,y)$, $x = g(s,t)$ and $y = h(s,t)$ are differentiable.

Then,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

partial derivatives.

Example

If $z = (x-y)^5$, $x = s^2 t$ and $y = st^2$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= 5(x-y)^4 \cdot 2st + -5(x-y)^4 t^2$$

$$= 5t(x-y)^4(2s-t)$$

and
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= 5(x-y)^4 \cdot s^2 - 5(x-y)^4 \cdot 2st$$

$$= 5s(x-y)^4(s-2t).$$

Polar coordinates

You already know the cartesian (x,y,z) coordinates system, but some functions are easier to explain in polar coordinates. These are for two dimensions.

Main idea: To replace x and y (i.e. movements along perpendicular axes), we move in terms of angle and distance.

r: distance from the origin

θ: angle between the positive x-axis.

Every point (x,y) can be rewritten as $(r\cos\theta, r\sin\theta)$.

Conversely, we can find (r,θ) by writing $r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

Here, r and θ are functions of the variables x and y.

Example

If x increases of 0.3 m starting at 2 meters, and y decreases from -1.5 m from 8m, what is the variation in r? in θ?

with the chain rule,

$$\frac{dr}{dt} = \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt}$$

That implies that

$$\frac{dr}{dt} = \left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\right)\Bigg|_{\substack{x=2\\y=8}} 0.3 + \left(\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\right)\Bigg|_{\substack{x=2\\y=8}} (-1.5)$$

$$= \frac{1}{\sqrt{4+64}}(0.6-12)$$

$$= \frac{-11.4}{\sqrt{68}}$$

So the variation in r is $\frac{-11.4}{\sqrt{68}} \approx -1.382$ m.

In θ,

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial x}\frac{dx}{dt} + \frac{\partial\theta}{\partial y}\frac{dy}{dt}.$$

$$= \left(\frac{-y}{\left(1+\frac{y^2}{x^2}\right)x^2}\right)\Bigg|_{\substack{x=2\\y=8}} 0.3 + \left(\frac{1}{x\left(1+\frac{y^2}{x^2}\right)}\right)\Bigg|_{\substack{x=2\\y=8}}(-1.5)$$

$$= \frac{-y}{x^2+y^2}\Bigg|_{\substack{x=2\\y=8}} \cdot 0.3 + \frac{1}{x+\frac{y^2}{x}}\Bigg|_{\substack{x=2\\y=8}}(-1.5)$$

$$= \frac{-8}{4+64} \cdot 0.3 + \frac{1}{2+32}(-1.5)$$

$$= \frac{1}{34}(-4 \cdot 0.3 - 1.5)$$

$$= \frac{-2.7}{34}$$

So the variation in θ is $\frac{-2.7}{34} \approx -0.079$ radians

Reference: James STEWART. Calculus, 8^{th} edition. § 14.5

Math 8
Winter 2020

Preliminary Homework
Assigned Monday, February 24
Due Wednesday, February 25

Note: Preliminary homework is always graded credit or no credit. **You get full credit for completing the assignment, whether or not your answers are correct.** The purpose of preliminary homework is to start you thinking about the topic of the next class.

You may use your preliminary homework in activities with your classmates. You should be sure to think about these questions so you will be prepared.

Preliminary homework is always due at the *beginning* of class.

Assignment: A drone flies above the garden of a prison. A wall is placed along the West–East axis, and a bird nest is placed on top of the wall. At time $t = 0$, the drone departs from the top of the wall, above the bird nest.

We look at the drone position with a satellite, which means everything looks flat. Let θ be the angle formed by the East part of the wall, the bird nest, and the drone. This is defined whenever the drone is not above the bird nest.

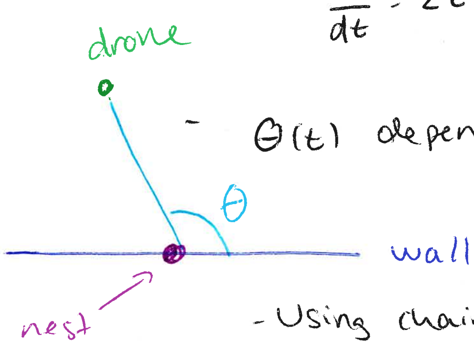
1. Assume at time t_0 (this is not the initial time), the position of the drone is 6 meters North of the wall, and 3 meters East of the nest. What is the value of the angle θ?

2. From this position, the drone flies m meters North. What is the new value of θ?

3. Between times $t = 0$ (seconds) and $t = 5$, the drone moved at a rate of $\frac{1}{1+t}$ m/s North, and $2t$ m/s East. At time $t = 4$, it was located 16 meters East of the bird nest, and $\ln(5)$ meters North of the wall. At time $t = 4$, how fast was the angle θ changing? This should be described by a derivative $\frac{d\theta}{dt}$.

For 3:

- express θ as a function of x and y, where y is the distance to the wall (North), and x is the distance East of the nest: $\theta = \arctan(y/x)$ if $x \neq 0$.
- x and y are themselves functions of time:
  - $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = \frac{1}{1+t}$. Also, $y(4) = \ln(5)$ and $x(4) = 16$.



- $\theta(t)$ depends on both x and y:

$$\theta(t) = \arctan\left(\frac{y(t)}{x(t)}\right).$$

- Using chain rule.

$$\frac{d\theta}{dt} = \frac{1}{1 + \frac{y(t)^2}{x(t)^2}} \cdot \left(\frac{1}{x(t)}\frac{dy}{dt} - \frac{y(t)}{x(t)^2}\frac{dx}{dt}\right)$$

product rule with $y(t)$ and $\frac{1}{x(t)}$

At time $t=4$, that is

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{\ln(5)}{16}\right)^2} \cdot \left(\frac{1}{16}\cdot\frac{1}{5} + \frac{\ln(5)}{16^2}\cdot 8\right)$$

$$= \frac{\frac{16}{5} + 8\ln(5)}{16^2+\ln(5)^2}$$

$$\approx 0.0621.$$

That means that, at time $t=4$, θ changes at a rate of ~ 0.0621 radians per second.