Math 8 - Lecture 25 Nadia Lafrenière

02/28/2020

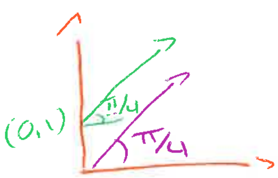
Directional derivative & Gradient vector

Reminder: The partial derivatives f_x and f_y are the derivatives of the function f in the direction of x- and y-axes, respectively.

How can we find the derivative in any direction?

Example

What is the derivative of $f(x,y) = y\cos(xy)$ in $(0,1)$ in the direction that makes an angle of $\pi/4$ counterclockwise with the positive x-axis?



The derivatives in $(0,1)$ in the direction of the main axes are the partial derivatives:

$$f_x(0,1) = (-y^2\sin(xy))\Big|_{\substack{x=0\\y=1}} = 0$$

$$f_y(0,1) = (-xy\sin(xy) + \cos(xy))\Big|_{\substack{x=0\\y=1}} = 1.$$

A suitable candidate: a weighted sum of f_x and f_y?

Our guess: it is $\sqrt{2}/2$ $\left(= 0\cdot\cos\left(\frac{\pi}{4}\right) + 1\cdot\sin\left(\frac{\pi}{4}\right)\right)$.

Definition

If f is a differentiable function of x and y, then f has a directional derivative in the direction of a unit vector $u = \langle a,b\rangle$ and

$$D_u f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\cdot b$$

$$= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle.$$

The vector function $\langle f_x(x,y), f_y(x,y)\rangle$ is called the gradient of f, denoted ∇f ("del f").

Example

Find the directional derivative of $f(x,y) = xy^3 - x^2$ at $(1,2)$ in the direction $\pi/3$, (with the positive x-axis, counterclockwise).

The direction $\frac{\pi}{3}$ can be represented by the unit vector

$\langle \cos(\frac{\pi}{3}), \sin(\frac{\pi}{3}) \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$.

The gradient is $\langle f_x(x,y), f_y(x,y) \rangle = \langle y^3 - 2x, 3xy^2 \rangle$.

The directional derivative is

$$\nabla f \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \frac{1}{2}(y^3 - 2x) + \frac{\sqrt{3}}{2}(3xy^2)$$
$$= \frac{1}{2}(y^3 - 2x + 3\sqrt{3}xy^2).$$

At $(1,2)$, this is $3 + 6\sqrt{3}$

Example

Find the directional derivative of $f(x,y) = e^x \sin(y)$ at $(0, \pi/3)$ in the direction of the vector $\langle -6, 8 \rangle$.

The vector $\langle -6, 8 \rangle$ is not a unit vector, but $\langle \frac{-3}{5}, \frac{4}{5} \rangle$ is and has the same direction.

$\nabla f(x,y) = \langle e^x \sin(y), e^x \cos(y) \rangle$ and $\nabla f(0, \pi/3) = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$.

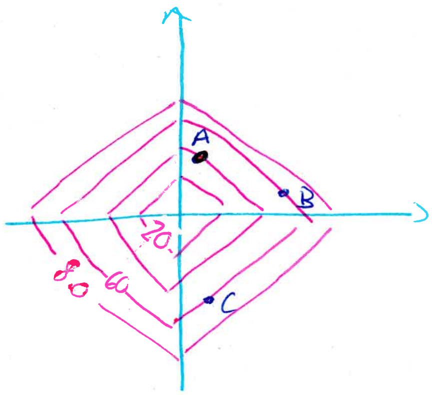
Hence, the directional derivative is $\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle \cdot \langle \frac{-3}{5}, \frac{4}{5} \rangle = \frac{-3\sqrt{3}}{10} + \frac{4}{10} = \frac{4 - 3\sqrt{3}}{10}$.

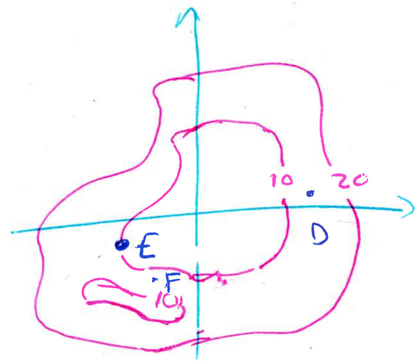
Proposition

The direction of ∇f is the direction in which f increases the fastest.

Example

Approximate the gradient's direction at A, B, C, D, E, and F.





A, B : $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

C : $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

D : $(1, 0)$

E : $(0, 1)$

F : one cannot tell
(F is between two parts of a level sets with the same value)

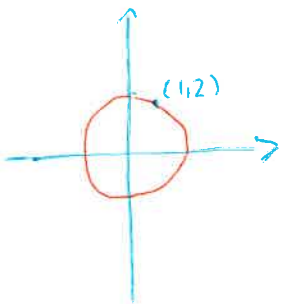
Proposition

At each point (a,b), the gradient $\nabla f(a,b)$ is perpendicular to the level set $f(x,y) = f(a,b)$.

This is because the level set gives the direction in which the function is not increasing, and the gradient gives the direction where it increases the most.

Example

Let $f(x,y)=x^2+y^2$. At $(1,2)$, $\nabla f(1,2)=(2,4)$.



Since f increases when (x,y) goes further away from $(0,0)$, it is not surprising that $\langle 1,2\rangle$ and ∇f have the same direction.

Example

What is the maximum rate of change of $f(x,y)=4y\sqrt{x}$ at $(4,1)$?

The direction of the maximum rate of change is the one of $\nabla f(4,1)$:

$$\nabla f(x,y)=\left\langle \frac{4y}{2\sqrt{x}}, 4\sqrt{x}\right\rangle = \left\langle \frac{2y}{\sqrt{x}}, 4\sqrt{x}\right\rangle \text{ and } \nabla f(4,1)=\langle 1,8\rangle.$$

A unit vector in that direction is $\left\langle \frac{1}{\sqrt{65}}, \frac{8}{\sqrt{65}}\right\rangle$

The maximum rate of change is $|\langle 1,8\rangle|=\sqrt{65}$, and this is in the direction of $\langle 1,8\rangle$.

Significance of the gradient vector: it is normal to a level curve (or, in higher dimension, to a level surface), the latter indicating a direction in which the function is not increasing.

Example

We proved that the tangent plane to $z=x^2+y^2$ at $(1,2,5)$ is $z=2x+4y-5$.

We can find it in another way by taking the level surface $z-x^2-y^2=0$, and find a normal vector with the gradient.

Let $g(x,y,z)=z-x^2-y^2$. Then $\nabla g(x,y,z)=\langle -2x,-2y,1\rangle$ and $\nabla g(1,2,5)=\langle -2,-4,1\rangle$.

Hence, $\langle -2,-4,1\rangle$ is normal to $z=x^2+y^2$, and $-2x-4y+z=5^*$ is the tangent plane to the paraboloid in $(1,2,5)$. ^{*5} here is found by replacing x,y and z.

Reference: James STEWART. Calculus, 8th edition. §14.6