Math 8 - Lecture 26

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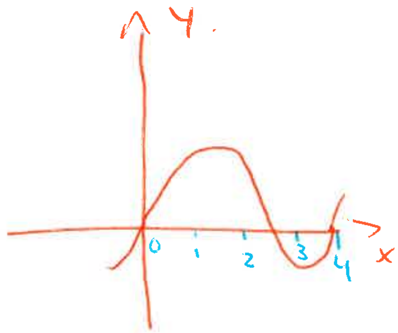
Maximum and minimum values

Example Let the graph below be the derivative of $f(x)$. Where is f maximal? minimal?

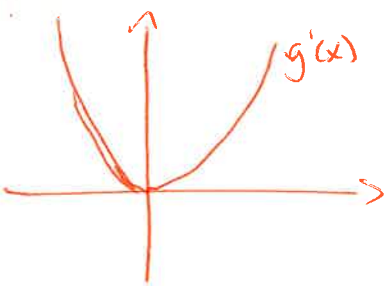
f:



The derivative has zeroes in 0, 2.5 and 4. They are the only places where minima or maxima can appear.

Here, 0 and 4 are local minima, and 2.5 is a maximum. (that can be seen by taking the integral).

g:



Where is $g(x)$ maximal? minimal?

The only candidate is in 0, since this is the only zero of the derivative. However, $g'(x)$ looks like x^2, so $g(x)$ must resemble $\frac{x^3}{3}$, that has no minimum, nor maximum.

Summary

For functions of one variable,

- If f has a minimum or a maximum in a, then $f'(a)=0$.
- It happens that $f'(a)=0$ and that a is not a minimum nor a maximum.

How can we generalize that to functions of two variables?

Definition

A function of two variables has a local maximum (respectively minimum) at (a,b) if $f(x,y) \le f(a,b)$ (resp. $f(x,y) \ge f(a,b)$) for all (x,y) near (a,b). The number $f(a,b)$ is a local maximum (resp. minimum) value.

If $f(x,y) \le f(a,b)$ for all (x,y) in the domain of f, then (a,b) is a global maximum

Theorem

If f has a local maximum or minimum at (a,b) and the first-order partial derivatives of f exist, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

A point (a,b) such that $f_x(a,b)=f_y(a,b)=0$ is a critical point. (It might not give rise to a minimum or a maximum).

Example

Find the minima and maxima of $f(x,y) = x^3-3x+3xy^2$.

The partial derivatives are

$$f_x(x,y) = 3x^2-3+3y^2 \quad \text{and} \quad f_y(x,y) = 6xy.$$

They are equal to zero when:

- $f_x(x,y)=0 \iff x^2+y^2=1$ (i.e. on the circle of radius 1 centered at the origin) and
- $f_y(x,y)=0 \iff x=0$ or $y=0$

Hence, the critical points are

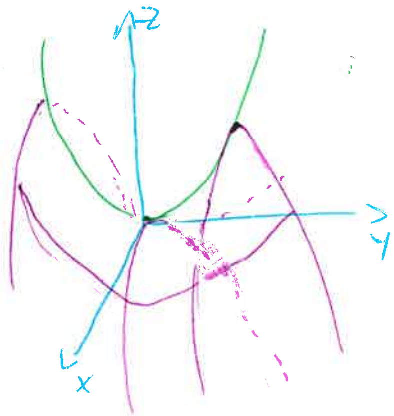
(a,b)	$f(a,b)$	min./max/other
$(0,1)$	0	nothing?
$(0,-1)$	0	nothing?
$(1,0)$	-2	minimum?
$(-1,0)$	2	maximum?

We lack information to tell wether or not it is a extremum. ③

Example

Find all the extrema of $f(x,y)=x^2-y^2$.

The only candidate is when $2x=-2y=0$, hence the origin.



However, if we look at the intersection with the yz-plane, the origin seems to be a maximum.

On the xz-plane, the origin seems to be a minimum.

Such a point is called a saddle point.

The following is a way to distinguish critical points.

Second Derivative test

Suppose the second partial derivatives of f are continuous near (a,b), and suppose (a,b) is a critical point (i.e. $f_x(a,b)=f_y(a,b)=0$). Let

$$D=D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then (a,b) is a local minimum.
(b) If $D>0$ and $f_{xx}(a,b)<0$, then (a,b) is a local maximum.
(c) If $D<0$, then (a,b) is not a minimum nor a maximum.

Notice that if $D=0$, one cannot say anything with this test.

D is called the discriminant.

Example

The origin for $f(x,y)=x^2-y^2$ fits in case (c):

$f_{xx}(x,y)=2$, $f_{yy}(x,y)=-2$, $f_{xy}(x,y)=0$

Example

For $f(x,y) = x^3 - 3y + 3xy^2$ (continued from page 2),

$$f_{xx}(x,y) = 6x, \quad f_{yy}(x,y) = 6x, \quad f_{xy}(x,y) = 6y.$$

Hence,

(a,b)	$D(a,b)$	min/max/other
$(0,1)$	-36	nothing
$(0,-1)$	-36	nothing
$(1,0)$	36	minimum
$(-1,0)$	36	maximum.

Geogebra?

Applications

This can be used to solve optimisation problems.

Example

The extrema of what function are you looking for if you want to have the shortest distance from the point $(-2,1,0)$ to the plane $x + 2y + z = 4$?

The distance from $(-2,1,0)$ to (x,y,z) is

$$d = \sqrt{(x+2)^2 + (y-1)^2 + z^2}.$$

On the plane $x+2y+z=4$, $z = 4-x-2y$. Hence, the distance from $(-2,1,0)$ to any point in that plane is

$$d = \sqrt{(x+2)^2 + (y-1)^2 + (4-x-2y)^2}$$

It will be minimal whenever its square will be (since $\sqrt{x}$ is a monotonic.)

To find the minimum of d^2, we calculate its partial derivatives:

$$d^2 = (x+2)^2 + (y-1)^2 + (4-x-2y)^2$$

$$d^2_x(x,y) = 2(x+2) - 2(4-x-2y) \text{ and } d^2_y(x,y) = 2(y-1) + 4(4-x-2y).$$

Solving for $d^2_x(x,y) = d^2_y(x,y) = 0$, we get

$$4x + 4y - 4 = 10y + 4x - 18 = 0.$$

And this happens only when $x = \frac{-4}{3}$, $y = \frac{7}{3}$ (and hence, $z = \frac{2}{3}$)

Because of the nature of the function, this is the minimum, and the minimum distance is $\sqrt{\left(\frac{-4}{3}+2\right)^2 + \left(\frac{7}{3}-1\right)^2 + \left(\frac{2}{3}\right)^2} = \frac{\sqrt{24}}{3} = \frac{2\sqrt{6}}{3}$

We can also find the extrema on a bounded domain D. If we want to know the minimum and maximum of a function over D:

(i) Find the value of f at its critical points in D.

(ii) Find the extreme values on the boundary of D.

(iii) The largest of these (from steps (i) and (ii)) is the absolute maximum value over D, and the smallest is the absolute minimum value.

Example

Find the extrema of $f(x,y) = xy^2$ over $D = \{(x,y) \mid x \geq 0, y \geq 0, x^2+y^2 \leq 3\}$

(i) The critical points satisfy $y^2 = 2xy = 0$, and this is the whole x-axis. Since f is nonnegative over D and 0 on the x-axis, f has a minimum on the x-axis.

(ii) On the y-axis, the function is also 0 and thus also a minimum.

- On $x^2+y^2=3$ (the other part of the boundary), the function xy^2 can be rewritten as $x(3-x^2) =: g(x)$.

$g(x)$ has critical points when $3-3x^2=0$, so in $x=1$. Since $g''(x)<0$, this is a maximum, and $f(1,\sqrt{2}) = 2$. is the absolute maximum of f over D. The minimum is 0 over the x- and y-axes.

Reference: James STEWART. Calculus, 8^{th} edition. §14.7.