Math 8 - Lecture 27

Lagrange multipliers

Nadia Lafrenière

3/4/2020

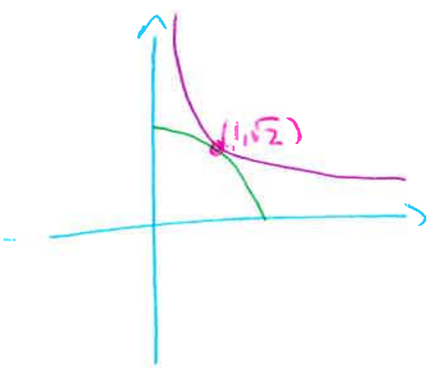
Recall the example from last class:

We were trying to maximize xy^2 with the constraints $x^2+y^2=3$, $x \geq 0$ and $y \geq 0$.

To solve it we used $y^2 = 3-x^2$, and replaced in the first equation.

However, the constraint can be more complicated, so this method does not always work.

Observation: At $(1,\sqrt{2})$ (i.e. the point where xy^2 is maximal), the level curve $xy^2=2$ is tangent to the circle $x^2+y^2=3$.



Algebraically: - the tangent to the circle is $\langle -\sqrt{2}, 1\rangle$.

- the tangent to $xy^2=2$ is the tangent to $y=\frac{\sqrt{2}}{\sqrt{x}}$, which is $y' = -\frac{\sqrt{2}}{2x^{3/2}}$.

In $x=1$, the tangent vector is $\langle 1, -\frac{\sqrt{2}}{2}\rangle$, which is parallel to $\langle -\sqrt{2}, 1\rangle$.

Reason: If it is not tangent, then the level curve crosses the curve for the constraint. That means that on one side of the level curve, the values are greater, and on the other side, they are lower. That contradicts the extremality of that point.

Theorem

If the function f has an extremum at (a,b,c) among the set of (x,y,z) satisfying the constraint $g(x,y,z)=k$ ($k \in \mathbb{R}$), then $\nabla f(a,b,c)$ and $\nabla g(a,b,c)$ are parallel.

Corollary

If $\nabla g(a,b,c) \neq \vec{0}$, then there exists a number λ such that

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c).$$

Method of Lagrange multipliers

To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$, (assuming that these values exist and $\nabla g \neq \vec{0}$).

(a) Find all values of x,y,z and λ such that

(i) $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

and (ii) $g(x,y,z)=k$.

(b) Evaluate f at all these points. The largest value is the maximum value of f; the smallest is the minimum.

Example

Find the extreme values (minima, maxima) of $f(x,y)=x^2-y^2$ subject to $x^2+y^2=1=g(x,y)$

(i) What is your intuition?

(ii) Find the gradients:

$$\nabla f(x,y) = \langle 2x,-2y\rangle \quad \text{and} \quad \nabla g(x,y) = \langle 2x,2y\rangle$$

(iii) They are parallel if

(i) $x=0$, and then $\lambda=-1$. Then, since $x^2+y^2=1$, $y=\pm1$.

or (ii) $y=0$, and then $\lambda=1$. Then, $x^2+y^2=1$ implies $x=\pm1$.

(iv) Find the values of f in these four points.

(x,y)	$f(x,y)$	min/max/other
$(0,1)$	-1	min
$(0,-1)$	-1	min
$(1,0)$	1	max
$(-1,0)$	1	max.

Example

Find the extremal values of $f(x,y,z) = 2x + 2y + z$, subject to $g(x,y,z) = x^2+y^2+z^2 = 9$.

Find the gradients.

$$\nabla f(x,y,z) = \langle 2,2,1\rangle \quad \text{and} \quad \nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle$$

Here, they are parallel vectors only if $x = y = 2z$. Hence, all the multiples of $\langle 2,2,1\rangle$ that satisfy $x^2+y^2+z^2=9$ are candidates.

They are the points $(2,2,1)$ and $(-2,-2,-1)$.

(x,y,z)	$f(x,y,z)$
$(2,2,1)$	9
$(-2,-2,-1)$	-9

The minimum is -9 and is reached in $(-2,-2,-1)$

The maximum is 9 and is reached in $(2,2,1)$.

Example

Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2+y^2 \le 1$.

What is your intuition?

According to the procedure presented last lecture, we must find the critical points inside the disk, and use the Lagrange multipliers on the boundary.

(i) Critical points

$f(x,y) = x^2+2y^2$ has only critical point $(0,0)$.

Since f is always nonnegative and $f(0,0) = 0$, f has a minimum in $(0,0)$.

(ii) On the boundary, define $g(x,y) = x^2 + y^2$ and require that $g(x,y) = 1$.

Then, find the gradients:

$$\nabla f(x,y) = \langle 2x, 4y \rangle \quad \text{and} \quad \nabla g(x,y) = \langle 2x, 2y \rangle,$$

and they are parallel when $x = 0$ or $y = 0$.

Hence,

(x,y)	$f(x,y)$	min/max/other
$(0,1)$	2	max
$(0,-1)$	2	max
$(1,0)$	1	min on the circle, but nothing on the disk.
$(-1,0)$	1	
$(0,0)$	0	min

circle

disk.

The minimum is 0 in $(0,0)$ and the maxima are 2 in $(0,1)$ and $(0,-1)$.

Reference: James STEWART. Calculus, 8th edition. § 14.8.