Math 8 - Lecture 3

Limit of Taylor series

Nadia Lafrenière
01/10/2020

A series is a sum of all the terms in a sequence. It is an infinite sum. Example: sequence $(a_n)_{n\in\mathbb{N}}$ <-> series $\sum_{k=0}^{\infty} a_k$

Up to now in the course, we have seen Taylor polynomials and limits of sequences.

Consider now the sequence $\{T_n\}_{n\in\mathbb{N}}$ of Taylor polynomials of a given function. We would like to know $\lim_{n\to\infty} T_n$.

Definition

The Taylor series for $f(x)$ centered at a is the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Its nth partial sum is the degree $n-1$ Taylor polynomial centered at a.
↳ Sum of the first terms.

Our hope is that the Taylor series for $f(x)$ is equal to the function $f(x)$. It is true in many nice cases.

Examples

- If f is a polynomial, its Taylor series is equal to f.
- We saw Wednesday that the Maclaurin polynomial for $f(x) = \frac{1}{1-x}$ is $T_n(x) = 1 + x + \dots + x^n$. We also saw that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, when $-1 < x < 1$.

Definition

The series $\sum_{n=0}^{\infty} a_n$ is a geometric series with ratio r if for every n, we have $\frac{a_{n+1}}{a_n} = r$ (for some number r).

Equivalently, $\sum_{n=0}^{\infty} a_n = a_0 \sum_{n=0}^{\infty} r^n$.

Proposition

If $\sum_{n=0}^{\infty} a_n$ is a geometric series with ratio r and first term a_0, then

$$\sum_{n=0}^{\infty} a_n = \begin{cases} \frac{a_0}{1-r} & |r|<1 \\ \text{divergent} & |r| \geq 1 \end{cases}$$

Example

Find to what $\sum_{k=0}^{\infty} \frac{5 \cdot 2^{k+2}}{3^{2k}}$ converges.

Solution

$$\sum_{k=0}^{\infty} \frac{5 \cdot 2^{k+2}}{3^{2k}} = 5 \cdot 2^2 \sum_{k=0}^{\infty} \frac{2^k}{(3^2)^k} = 5 \cdot 2^2 \cdot \sum_{k=0}^{\infty} \left(\frac{2}{9}\right)^k$$

$$= \frac{20}{1-\frac{2}{9}} = \frac{180}{7}$$

Example

Does $\sum_{k=0}^{\infty} \frac{5 \cdot 2^{2k}}{3^k}$ converge?

The answer is no, since this is $5 \cdot \sum_{k=0}^{\infty} \left(\frac{4}{3}\right)^k$, and $\frac{4}{3} > 1$.

Proposition (Comparison test)

Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are non-negative series (i.e. series with no negative term). If $0 \le b_n \le a_n$ for all n, then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \Rightarrow \sum_{n=0}^{\infty} b_n \text{ converges.}$$

Sort of a version of squeeze theorem.

You can combine this proposition with the rules in the appendix.

Example

If you know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (it does, we will see it next week), you can prove using the comparison test and the tail end rule that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges:

For $n \ge 4$, $n! > n^2$. Then, $0 < \frac{1}{n!} < \frac{1}{n^2}$ for $n > 4$.

By comparison test, $\sum_{n=4}^{\infty} \frac{1}{n!}$ converges, since $\sum_{n=4}^{\infty} \frac{1}{n^2}$ does.

By tail-end rule, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges, since this is $1 + \frac{1}{2} + \frac{1}{3} + \sum_{n=4}^{\infty} \frac{1}{n!}$.

Absolute convergence

Definition

The series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

⚠ Caveat: $\sum_{n=0}^{\infty} |a_n| \neq \left|\sum_{n=0}^{\infty} a_n\right|$.

Proposition

If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Caveat: The converse is false.

For example, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$

is convergent (by the criterion we will see soon), but it is not absolutely convergent, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Alternating series

An alternating series is a series that alterns between positive and negative terms (like $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots$ above).

Proposition (Alternating series test)

If a series $\sum_{n=0}^{\infty} a_n$ satisfies the following three conditions, then it converges:

(1) The terms a_n alternate between positive and negative.

(2) The terms a_n are decreasing in absolute value, i.e. $|a_{n+1}| \leq |a_n|$ for all n.

(3) The terms a_n are approaching 0, that is $\lim_{n\to\infty} a_n = 0$.

Example

The Maclaurin series for $f(x) = \cos(x)$ converges for all values of x.

$$\lim_{n \to \infty} T_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

- If $x = 1$, it is clear that all three conditions are satisfied.
- If $|x|$ is big, we can use the tail end rule, since except for the first few terms, the terms will decrease, making the second condition true.

Proposition

The Taylor series for $\sin(x)$ and $\cos(x)$ converge to $\sin(x)$ and $\cos(x)$, respectively, for all values of x.

This proposition works independently of where it is centered, i.e. for all values of a.

Proposition

The Taylor series for e^x converge to e^x, for all values of x and a.

Example

The Maclaurin series for e^x is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

In $x=1$, it means that $\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = e$.