Math 8 - Lecture 4

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Approximations and error bounds

Last lecture, I claimed that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

(or, in fact, that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$).

How far from e is $1+1+\frac{1}{2}+\frac{1}{6}$? Is that equivalent to $\sum_{n=4}^{\infty} \frac{1}{n!}$?

- It is, because we know that $\sum_{n=0}^{\infty} \frac{1}{n!} = e$.
- Using a calculator, we get that $\left| e - \left(1+1+\frac{1}{2}+\frac{1}{6}\right)\right| \approx 0.052$.
- Today, we see how to bound $\sum_{n=4}^{\infty} \frac{1}{n!}$ without a calculator. We see two techniques: one for alternating series, and one with comparison.

When we estimate a smooth function f using a Taylor approximation $T_n(x)$, the <u>error</u> is the difference between the estimate and the actual value $|T_n(x) - f(x)|$. Of course, we are interested in that value, since it determines if the approximation is good.

Alternating series

Recall from last lecture that $\sum_{n=0}^{\infty} a_n$ is an alternating series if

(i) the terms alternate between positive and negative.

(ii) $|a_{n+1}| \leq |a_n|$ for all n

(iii) $\lim_{n\to\infty} a_n = 0$

Proposition

If $\sum_{n=0}^{\infty} a_n$ is an alternating series, then $\left|\sum_{n=0}^{\infty} a_n\right| \leq |a_0|$.

Using the tail-end property, we get the following.

Corollary

If $\sum_{n=0}^{\infty} a_n$ is an alternating series then $\left|\sum_{n=m}^{\infty} a_n\right| \leq |a_m|$

Example

Let $T_4(x)$ be the 4th Taylor polynomial for $f(x)=\cos(x)$ in $a=0$.

Then, the error for estimating $\cos(0.05)$ is bounded by

$$\left|\sum_{n=3}^{\infty} \frac{(-1)^n x^{2n}}{2n!}\right| \leq \frac{x^6}{6!} = \frac{0.05^6}{6!} \approx 2.17\times 10^{-11}.$$

This is because the MacLaurin series for f is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} \quad \text{(see last lecture)},$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} - T_4(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} - \underbrace{\frac{(-1)^0 x^0}{0!}}_{\text{"0-th" term}} - \underbrace{\frac{(-1)^1 x^2}{2!}}_{\text{1st term}} - \underbrace{\frac{(-1)^2 x^4}{4!}}_{\text{2nd term}}.$$

of the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$

Question: How big must n be so $|T_n(-1) - e^{-1}| < 0.05$, where T_n approximates e^x?

Solution

The Maclaurin series for e^x, is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, and I claimed last class that it converges to e^x for all values of x.

- $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ is alternating.

So we know that

$$|e^{-1} - T_n(-1)| = \left|\sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}\right| \le \left|\frac{(-1)^{n+1}}{n+1!}\right| = \frac{1}{(n+1)!}$$

To find for what value of n, $\frac{1}{(n+1)!} < 0.05$, we write 0.05 as $\frac{1}{20}$. It turns out that

$$\frac{1}{4!} = \frac{1}{24} < \frac{1}{20} < \frac{1}{6} = \frac{1}{3!}.$$

Hence, $n+1 = 4$ and $|T_3(-1) - e^{-1}| < 0.05$

Geometric series

Question: What is $\sum_{n=5}^{\infty} \left(\frac{1}{3}\right)^n$?

Solution:

That can be rewritten as $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+5} = \frac{1}{3^5} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{3}{2} \cdot \frac{1}{3^5} = \frac{1}{2 \cdot 3^4}$.

That is also the value of $\left|\frac{1}{1-\frac{1}{3}} - 1 - \frac{1}{3} - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^4\right|$.

Comparison test

What happens if the series is neither alternating nor geometric?

In some cases, one can use the comparison test.

Recall that this means that if $0 \le a_n \le b_n$ for all n and $\sum_{n=0}^{\infty} b_n = B$, then $0 \le \sum_{n=0}^{\infty} a_n \le B$.

Question: what is the smallest n for $|e - T_n(1)| < 0.005$, if T_n approximates $f(x) = e^x$?

Solution

That is when

$$\left| \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} \right| = \left| \sum_{k=n+1}^{\infty} \frac{1}{k!} \right|.$$

But we know that $\frac{1}{k!}$ is positive and, for all $k > n+1$,

$$\frac{1}{k!} < \frac{1}{(n+1)^{k-n}} \frac{1}{n!}.$$

So the error is at most

$$\sum_{k=n+1}^{\infty} \frac{1}{n!} \frac{1}{(n+1)^{k-n}} = \frac{(n+1)^n}{n!} \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^k}$$

$$= \frac{(n+1)^n}{n!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^{k+n+1}}$$

$$= \frac{\cancel{(n+1)^n}}{n!\,(n+1)^{n+1}} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k}$$

$$= \frac{1}{n!\,(n+1)} \frac{1}{1-\left(\frac{1}{n+1}\right)} = \frac{1}{n!\,n}$$

We thus need to find the smallest n so $\frac{1}{n!\,n} < 0.005$, which is $n = 5$.