Math 8 - Lecture 5

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01/15/2020.

Power series

Today, we keep looking at convergence of series, but this time, we will consider series like functions, allowing differentiation and integration.

More precisely, we

- present one more test for convergence (ratio test)
- learn about radius of convergence of series

and - within radius of convergence, we differentiate and integrate the series, regarded as functions

Theorem (Ratio test)

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series $\sum_{n=0}^{+\infty} a_n$ converges.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ (or if it is infinite), the series $\sum_{n=0}^{\infty} a_n$ diverges.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

Example (Geometric series)

Using the ratio test, we can prove that geometric series $\sum_{n=0}^{\infty} r^n$ converges if and only if $-1 < r < 1$.

Example

Does the series $\sum_{n=0}^{\infty} \frac{n^3}{3^n}$ converge?

2 min
by themselves

Using the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right| = \lim_{n\to\infty} \frac{1}{3} \left|\frac{(n+1)^3}{n^3}\right|$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3$$

$$= 1/3 < 1$$

That series converges.

Example

Does the series $\sum_{n=1}^{\infty} 1/n^2$ converge?

1 min

Using the ratio test:

$$\lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1.$$

The test is inconclusive. (However, the answer is yes).

Convergence of power series

Definition

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

coefficients

variable x

More generally, a power series centered at a is

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots$$

Example

Geometric series, when all coefficients are 1, are evaluations of $\sum_{n=0}^{\infty} x^n$ in $x=r$.

We then say that the power series $\sum_{n=0}^{\infty} x^n$ converges when $|x|<1$, or in a radius 1 from 0.

Theorem

Convergence of power series.

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only 3 possibilities

- The series converges only at $x=a$.
- The series converges for all $x \in \mathbb{R}$.
- there is a positive number R such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$. Such a number R is the radius of convergence, and the interval of convergence, is the set of all values of x for which the series converges.

Examples

- Geometric series

  $\sum_{n=0}^{\infty} x^n$ has a radius of convergence of 1, and its interval of convergence is $(-1,1)$.

- The series $\sum_{n=0}^{\infty} n!\, x^n$ is a power series that converges only for $x=0$, and has thus a radius of convergence of 0.

- $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ has a radius of convergence of 1 (that can be seen through to the ratio test), and an interval of convergence of $[2,4)$

- $\sum_{n=1}^{\infty} \frac{x}{n!}$ has a radius of convergence of size ∞, and its interval of convergence is $(-\infty,\infty)$.

Reference: James STEWART. Calculus, 8th edition.
Sections 11.2, 11.6 and 11.8.

How do we find power series?

- When they are not hard to compute, Taylor series are great.
- Sometimes, derivatives and integrals can be more useful.

Theorem (term-by-term differentiation and integration)

If the power series $\sum_{n\geq 0} c_n (x-a)^n$ has radius of convergence $R>0$, the function f defined by

$$f(x) = \sum_{n\geq 0} c_n (x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$, and

(i) $f'(x) = c_1 + 2c_2 (x-a)^2 + 3c_3 (x-a)^3 + \dots = \sum_{n\geq 1} n c_n (x-a)^{n-1}$

(ii) $\int f(x)\, dx = C + c_0 (x-a) + c_1 \frac{(x-a)^2}{2} + \frac{c_2 (x-a)^3}{3} + \dots$

$$= C + \sum_{n\geq 0} c_n \frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of the power series in (i) and (ii) are both R.

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Question: What is a power series for $\ln(x)$, $0<x<2$?

Solution: We know that $(\ln(x))' = \frac{1}{x} = \frac{1}{1-(1-x)}$.

We also know that $\frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n$ when $|1-x|<1$

(i.e. when $0<x<2$)

Hence, by the term-by-term differentiation theorem,

$$\ln(x) = \int \frac{1}{x}\,dx$$

$$= \int \sum_{n=0}^{\infty} (1-x)^n\,dx$$

$$= \sum_{n=0}^{\infty} \int (1-x)^n\,dx \quad \text{when } 0<x<2.$$

$$= \sum_{n=0}^{\infty} \frac{-(1-x)^{n+1}}{n+1} + C.$$

To find the value of C, we look at one precise value of x (here $x=1$):

$$\ln(1) = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{-(1-1)^{n+1}}{n+1} + C = C,$$

which means $C=0$.

Question: Can you find the 7th Taylor polynomial of $\underbrace{\tan^{-1}(x)}_{\text{or } \arctan(x)}$ at $a=0$?

Doing the regular way for Taylor polynomials, we would get the following:

$$f(x) = \tan(x)$$

$$f'(x) = \frac{1}{1+x^2} \quad *$$

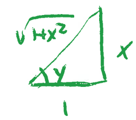
$$f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$f'''(x) = \frac{-2(1+x^2)-4x}{(1+x^2)^3}$$

$$\vdots$$

Explanation of $f'(x) = \frac{1}{1+x^2}$:

$$\arctan(x) = y$$
$$\Leftrightarrow \quad x = \tan(y)$$
$$\Rightarrow \quad (x)' = (\tan(y))'$$
$$\Rightarrow \quad 1 = \sec^2(y) \cdot y'$$
$$\Rightarrow \quad 1 = (1+x^2) \cdot y'$$
$$\Rightarrow \quad y' = \frac{1}{1+x^2}.$$



(deduced from $x = \tan(y)$.)

Of course, many things cancel each other out, but it still seems painful! And we are not still close to the 7th Taylor polynomial!

Using the theorem,

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{n\geq 0}^{\infty} (-x^2)^n \,dx \qquad \text{because this is a geometric series.}$$

$$= \int (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots)\,dx \qquad \text{expansion}$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \dots \qquad \text{term-by-term integration.}$$

To find C, we evaluate in $x=0$: $\tan(x)=0 \Rightarrow C=0$, in this case

Therefore,

$$\tan^{-1}(x) = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{2n+1},$$

in the interval of convergence of $\frac{1}{1+x^2}$, which is $(-1,1)$ (it is geometric).