Math 8 - Lecture 6

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Integrals as volumes and areas

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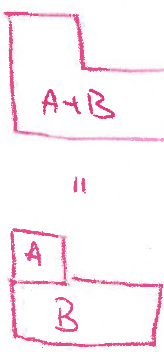
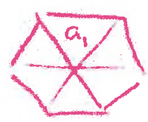
Last class

- we differentiated and integrated some sums, by considering the integral as the antiderivative.

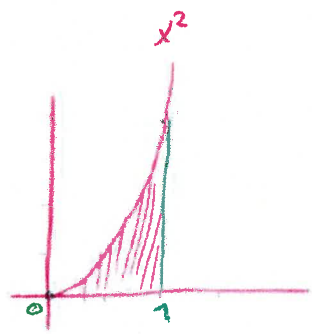
For this class, instead of using integrals to solve some sum's problems, we will define the integral as the limit of a given sum.

Question: How do we compute the area of a definite region?





a regular hexagon can be cut into regular triangles

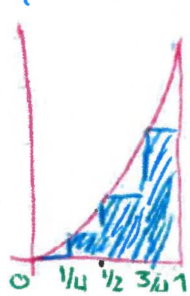


?

Can we decompose the area underneath a curve into rectangles/triangles?

Example: dividing the area below x^2 between 0 and 1 into 4 rectangles:

Ⓐ



Ⓑ

What are the problems?

- Ⓐ is too small, but Ⓑ is too high

Computing the estimated area:

(A)

height of 1st rectangle

width of the 1st retangle

$$\frac{1}{4}(0)^2+\frac{1}{4}\left(\frac{1}{4}\right)^2+\frac{1}{4}\left(\frac{2}{4}\right)^2+\frac{1}{4}\left(\frac{3}{4}\right)^2=\frac{7}{32}$$

(B)

$$\frac{1}{4}\left(\frac{1}{4}\right)^2+\frac{1}{4}\left(\frac{2}{4}\right)^2+\frac{1}{4}\left(\frac{3}{4}\right)^2+\frac{1}{4}1^2=\frac{15}{32}.$$

How can we improve?

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* We took the lowest (respectively highest) value of x^2 in a given interval to compute the area in (A) (resp. (B)). Could we take something in the middle? This is however not so well defined if the function is not monotone.

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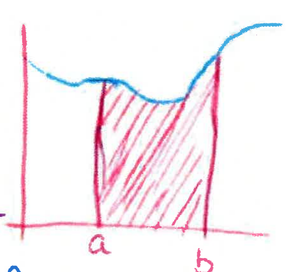
* If we take more rectangles, the estimate should be more precise.

Riemann sums

To compute the area under a function f, over an interval $[a,b]$:

1- Divide $[a,b]$ into many segments of width $\frac{b-a}{n}$.



2- In the i-th interval, choose a point x_i^* any point

3- For each interval, draw a rectangle of width $\frac{b-a}{n}$ such that the i-th is of height $f(x_i^*)$

4- Approximate the total area by the sum of the areas of the rectangle. The latter is called the Riemann sum

If all x_i^* are taken on the left (respectively right) end of the interval, the sum is called the left Riemann sum (resp. right Riemann sum), and we denote it x_i.

Theorem

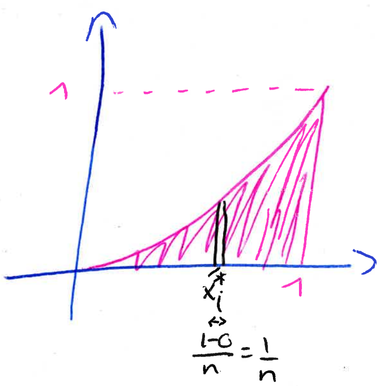
If the function f is integrable on $[a,b]$, the limit of the left and right Riemann sums are the same, and

$$\lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\left(\frac{b-a}{n}\right)}_{\substack{\text{width}\\\text{of a}\\\text{rectangle}}} \underbrace{f(x_k)}_{\substack{\text{height of}\\\text{the kth}\\\text{rectangle}}} = \int_a^b f(x)\,dx.$$

This is the <u>definite integral</u>.

Example

Estimating the area under $f(x)=x^2$ between 0 and 1:



1. We divide it into n many rectangles, each of width $\frac{1}{n}=\Delta x$. Each has height $(x_i^*)^2$, where x_i^* is a point in the i-th interval.
2. The total area is

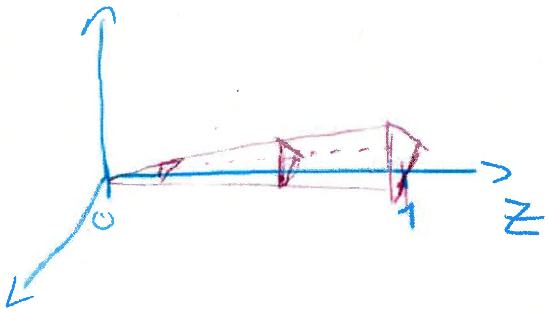
$$\sum_{i=1}^{n} (x_i^*)^2 \Delta x = \sum_{i=1}^{n} (\text{area of the i-th rectangle})$$

3. As $n\to\infty$, that is

$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$

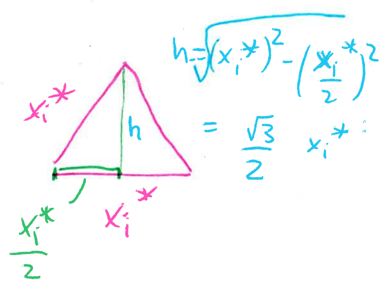
- Notice that Δx, the width of each rectangle, becomes dx. This is because as $n\to\infty$, $\Delta x = \frac{1}{n}$ goes infinitesimal.
- x_i^*, that is any point in the ith interval, becomes x, since the interval becomes so thin it goes to a single point.

Problem: Use the technique of the Riemann sum to compute the volume of the tetrahedron that is such that the projection in any $z\in[0,1]$ is an equilateral triangle of base z.



We cut the [0,1]-interval on z into n slices that are like thin triangular prisms, of thickness $\frac{1}{n}$.

If the point x_i^* is in the i-th interval, then the triangle has base x_i^*. Its area is $\frac{\sqrt{3}(x_i^*)^2}{4}$ (see picture on the right).



- The volume of the i-th prism is

$$\underbrace{\frac{\sqrt{3}(x_i^*)^2}{4}}_{\text{area}} \cdot \underbrace{\Delta x}_{\text{thickness}}$$

- The total volume of the tetrahedron is

$$\sum_{i=1}^{n} \frac{\sqrt{3}(x_i^*)^2}{4}\Delta x,$$ and as $n \to \infty$, this is $\int_0^1 \frac{\sqrt{3}}{4}x^2\,dx = \left(\frac{\sqrt{3}}{12}x^3\right)\Big|_0^1 = \frac{\sqrt{3}}{12}$.

See animation with Geogebra.

(Animation with Geogebra)

Definition

Let S be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, follows a continuous function $A(x)$, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\left(\frac{b-a}{n}\right) = \int_a^b A(x)\,dx$$

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x_i^* is in the i-th interval.

Average value

Question: What is the average value of x^2 between 0 and 1?

* We cannot take $\left(\frac{1-0}{2}\right)^2$, because that does only take into account one point.
* We can however take the area, and divide by the width.

Definition

The average value of an integrable (and therefore continuous) function on $[a,b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example

The average value of x^2 on $[0,1]$ is $\frac{1}{1-0}\left(\frac{1^3}{3}-\frac{0^3}{3}\right) = \frac{1}{3}$.

one more example.

Example

The mass density of a wire occupying the portion of the x-axis for $0 \le x \le 4\pi$ is $f(x) = \cos^3(x)\sin^3(x) + 2$. kg/m

What is the average mass density between 0 and 4π?

Solution

Using the average value theorem, the average mass density is

$$\frac{1}{4\pi}\int_0^{4\pi} \cos^3(x)\sin^3(x) + 2 \, dx = \frac{1}{4\pi}\int_0^{4\pi} \sin(x)\cos^3(x)(1-\cos^2(x)) + 2 \, dx$$

$$= \frac{1}{4\pi}\int_0^{4\pi} \sin(x)\cos^3(x) - \sin(x)\cos^5(x) + 2 \, dx$$

$$= \frac{1}{4\pi}\left(-\frac{\cos^4(x)}{4} + \frac{\cos^6(x)}{6} + 2x\right)\Big|_0^{4\pi}$$

$$= \frac{1}{4\pi}\left(\cancel{-\frac{\cos^4(4\pi)}{4}} + \cancel{\frac{\cos^4(0)}{4}} + \cancel{\frac{\cos^6(4\pi)}{6}} - \cancel{\frac{\cos^6(0)}{6}} + 8\pi - 0\right)$$

$$= \frac{8\pi}{4\pi} = 2.$$

The average mass density is then 2 kg/m.

References: James STEWART. Calculus, 8th edition.
§ 4.1, 4.2, 5.2 and 5.5.