Math 8 - Lecture 7

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Integration by parts

Integration by parts is a technique of integration that allows you to integrate products of integrals.

Theorem (Formula for integration by part).

Let f and g be differentiable functions.

Then,

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int g(x)\, f'(x)\, dx$$

Tip: This formula is easier to remember after doing the substitutions $u = f(x)$, $du = f'(x)dx$ and $v = g(x)$, $dv = g'(x)\,dx$.

Then, it becomes

$$\int u\, dv = uv - \int v\, du.$$

Proof of the theorem

We know the differentiation rule for the product

$$(f(x)\, g(x))' = f'(x) g(x) + f(x)\, g'(x)$$

Integrating on both sides, we get

$$f(x)\, g(x) = \int \left(f'(x) g(x) + f(x)\, g'(x)\right) dx$$

$$= \int f'(x) g(x) dx + \int f(x)\, g'(x)\, dx$$

Manipulating this equation, we get

$$\int f(x)\, g'(x) dx = f(x) g(x) - \int f'(x)\, g(x)\, dx$$

Examples

(i) $\int \ln(x)\,dx$

We can rewrite it as

$$\int \underbrace{\ln(x)}_{u} \cdot \underbrace{1\,dx}_{dv}.$$

$u = \ln(x)$ $\quad v = x$

$du = \frac{1}{x}dx$ $\quad dv = 1\,dx$

Integrating by part, this is

$$x\ln(x) - \int \cancel{x}\frac{1}{\cancel{x}}\,dx = x\ln(x) - x + C.$$

Does that make sense? Try differentiating

$x\ln(x) - x + C$.

Note: to know which part is u and which part is dv, keep in mind that we need to integrate $v \cdot du$. Usually, we look for a simpler function to integrate. It is not the case above, but, in general, we choose the polynomial for u whenever dv is some other function (like e^x or a trigonometric function).

(ii) $\int \underbrace{x}_{u}\; \underbrace{\sin(x)\,dx}_{dv}$

$u = x$ $\quad v = -\cos(x)$

$du = dx$ $\quad dv = \sin(x)\,dx$

is also, by integration by parts,

$$-x\cos(x) + \int \cos(x)\,dx = -x\cos(x) + \sin(x) + C.$$

(iii) $\int e^x \sin x \, dx$

Neither e^x nor $\sin x$ becomes simpler after differentiation, so we can choose either to be u and dv.

$$\int \underbrace{e^x}_{u} \underbrace{\sin x \, dx}_{dv} \qquad \begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned} \qquad \begin{aligned} v &= -\cos x \\ dv &= \sin x \, dx \end{aligned}$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

We can do integration by part again to integrate $e^x \cos x$.

Caveat: If you choose this time $e^x dx$ to be dv and $\sin x$ to be du, you will just get $\int e^x \sin x \, dx = \int e^x \sin x \, dx$. You must stick with the choice you made for which of dv and u is the trigonometric function and which is the exponential.

So

$$\int e^x \sin x \, dx = -e^x \cos x + \int \underbrace{e^x}_{u} \underbrace{\cos x \, dx}_{dv} \qquad \begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned} \qquad \begin{aligned} v &= \sin x \\ dv &= \cos x \, dx \end{aligned}$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

Adding $\int e^x \sin x dx$ on both sides:

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + C.$$

Dividing by 2, we find the solution

$$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x + C}{2}.$$

(iv) What is $\int \arctan(x)\,dx$?

By parts, with $u = \arctan(x)$, $dv = dx$.

$u = \arctan(x)$ $\quad V = x$

$du = \frac{1}{1+x^2}dx$ $\quad dv = dx.$

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

To integrate $\frac{x}{1+x^2}dx$, apply the substitution

$u = 1+x^2$

$du = 2x\,dx$

Thus,

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{1}{2}\,\frac{2x\,dx}{1+x^2}$$

$$= x\cdot\arctan(x) - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= x\arctan(x) - \frac{1}{2}\ln(u) + C$$

$$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$$

Does that make sense? Differentiate it to get $\arctan(x)$.

Reference: James STEWART. Calculus, 8th edition.
Section 7.1