Math 8 - Lecture 8

Volumes of revolution

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Summary of the lecture:

* Last class, we introduced the volume as an integral of the area of the cross-sections of a solid.
* This class is dedicated to computing the volume of solids that have a symmetry axis (such as spheres, cones, ...), through revolution around that axis.
* We see 3 methods: disks, washers and cylindrical shells are ways to approximate the volume.

Disks

* The disk method is useful when the solid touches the rotation axis.

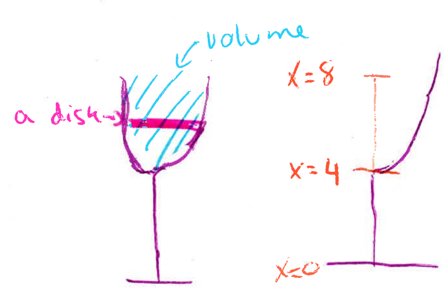
The method consists of dividing the solid into disks perpendicular to the rotation axis, and centered on it.

1. Divide the solid into the disks
2. Compute the area of each disk.
3. Integrate along the rotation axis.

Example

A glass of wine stands in front of a cartesian plane (!) such that its stand is in front of the y-axis. The stand is 4 inches high, and the glass is 8 inches high. If the bulb is shaped following $x^2+4=f(x)$, what is its volume?

- Disks centered around $x=0$ (y-axis)
- For $4\leq y\leq 8$, the radius of the disk is $\sqrt{y-4}$, thus the area is $\pi(y-4)$.



(2)

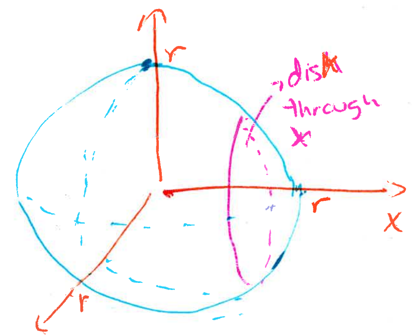
The volume of the glass is computed by

$$\int_4^8 \pi(y-4)\,dy = \pi\int_4^8 y-4\,dy = \pi\left(\frac{y^2}{2}-4y\right)\Bigg|_{y=4}^{y=8} = 8\pi,$$

and is thus 8π inches^3.

__Question__. Can you compute the volume of a sphere of radius r?

2 min.



* Around the x-axis.
* At a given value of x, the disk in the cross-section has radius $\sqrt{r^2-x^2}$ *
* We integrate for $-r \le x \le r$. The area at a given x is $\pi(r^2-x^2)$.

* because the projection of the sphere on the plane xy is a circle.

$$\int_{-r}^{r} \pi(r^2-x^2)\,dx = \pi\int_{-r}^{r} r^2-x^2\,dx = \pi\left(r^2x-\frac{x^3}{3}\right)\Bigg|_{-r}^{r} = \pi\left(\left(r^3-\frac{r^3}{3}\right)-\left(-r^3+\frac{r^3}{3}\right)\right)$$

$$= \frac{4\pi r^3}{3}$$

__Washers__

* It is similar to disks, but used when the solid is not touching the axis.
* A __washer__ is the difference of two disks, and its area is the difference of the areas of the disks.

__Example__

A thick salad bowl has a circular base of radius two inches, and then is straight inside and curly outside, according to

$y=(x-2)^2$ outside

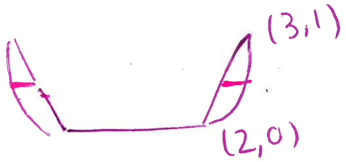
$y=x-2$ inside

and is rotating around the y-axis.

It is one inch high.

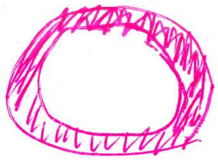
If the base is very thin (ie. negligeable), what is the volume of the bowl (the material)

We cut it into washers of outer radius $x = \sqrt{y} + 2$ and inner radius $x = y + 2$.



We integrate for $0 \le y \le 1$:

top view:



$$\int_0^1 \pi(\sqrt{y}+2)^2 - \pi(y+2)^2 \, dy$$

$$= \pi \int_0^1 -y^2 - 3y + 4\sqrt{y} \, dy$$

$$= \pi \left(-\frac{y^3}{3} - \frac{3y^2}{2} + 4 \cdot \frac{2}{3} y^{3/2} \right) \Big|_0^1$$

$$= \pi \left(-\frac{1}{3} - \frac{3}{2} + 4 \cdot \frac{8}{3} \right) = \frac{5\pi}{6},$$

and the volume of the salad bowl is $\frac{5\pi}{6}$.

Cylindrical shells

* The approximation is made by concentric cylinders.
* It is useful when the computations are hard using disks.
* The cylinders are around the symmetry axis.

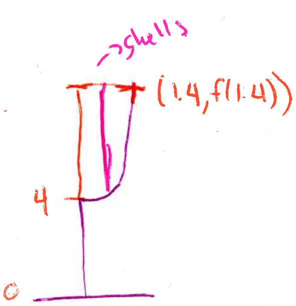
Example In class, I did instead the example on page 5.

If the glass from before has instead the shape of $f(x) = \tan(x)/x + 3$ (this is more realistic...) and the bulb has biggest radius 1.4 inch, what is the volume it contains?

* try to do it using the disks. It is hard!

Use the method of the cylindrical shells to compute it.

* Each cylinder has height $\ell - \frac{\tan(x)}{x} - 3$, where $\ell = f(1.4)$ is the height of the glass.
* We multiply by the circumference of the border of the cylinder: $2\pi x$
* the cover of the cylinder (the shell) has area $2\pi x \left(\ell - \frac{\tan(x)}{x} - 3\right)$



* We integrate over x, between 0 and 1.4.

$$\int_0^{1.4} 2\pi x\left(\frac{\ell - \tan(x)}{x} - 3\right)dx = \int_0^{1.4} 2\pi x\ell - 2\pi\tan(x) - 6\pi x\, dx$$

$$= \left(\cancel{2}\pi\ell\frac{x^2}{\cancel{2}} + 2\pi\ln(\cos(x)) - 3\pi x^2 \right|_0^{1.4}$$

$$= \pi \underbrace{f(1.4)}_{\ell = \frac{\tan(1.4)}{1.4} + 3} \cdot (1.4)^2 + 2\pi \cdot \ln(\cos(1.4)) - 3\pi(1.4)^2$$

$$= \pi \cdot \tan(1.4)(1.4) + \cancel{3\pi(1.4)^2} + 2\pi\ln(\cos(1.4)) \cancel{- 3\pi(1.4)^2}$$

and the volume of the glass is $(\pi\tan(1.4)\cdot(1.4) + 2\pi\ln(\cos(1.4)))$ inches^3.

General method:

1- Divide the area of the rotating part in segments parallel to the axis of rotation.

2- Create a cylindrical shell by rotating this segment. Its area is $2\pi hx$, where h is the height of the cylinder at x, if we are rotating along the y-axis.

3- Integrate on the x-axis (perpendicular to the rotation).

Disks or shells

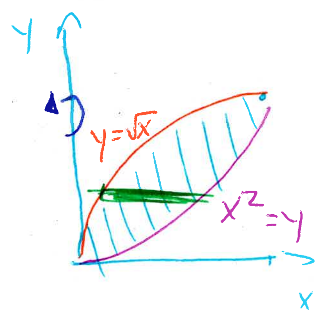
We can in general use either method. The best way to know which one should be used is to try to compute the volume.

Reference: James STEWART. Calculus, 8th edition. Sections 5.2 and 5.3

Example (volume by washers)

Find the volume of rotation around the y-axis of the area between $y=x^2$ and $y=\sqrt{x}$, for x between 0 and 1

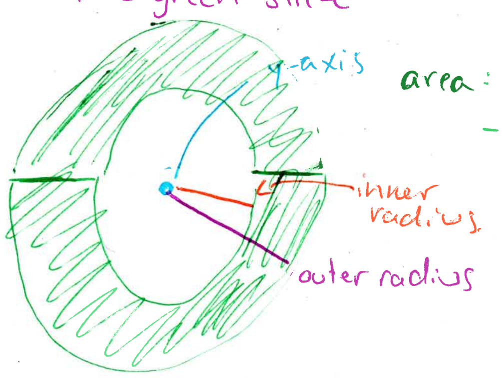
1. Draw a picture



Between $x=0$ and $x=1$ (but also between $y=0$ and $y=1$):

- $\sqrt{x}=y \Rightarrow x=y^2$ inner radius
- $x^2=y \Rightarrow x=\sqrt{y}$ outer radius

top view, after rotation, of the green slice



area: $\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$

The total volume is obtained by integrating the area of each slice over the interval [0,1] on the y-axis:

$$\int_0^1 \pi\left((\sqrt{y})^2-(y^2)^2\right)dy = \int_0^1 \pi(y-y^4)\,dy$$

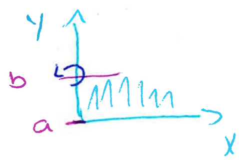
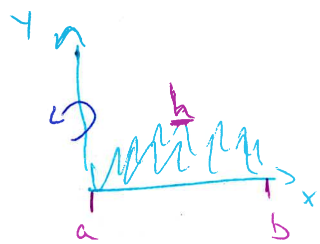
$$= \pi\left(\frac{y^2}{2}-\frac{y^5}{5}\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$= \frac{3\pi}{10}$$

The volume of revolution is $3\pi/10$ cubic units.

Recapitulation

Axis around which we revolve	y-axis	x-axis
Washers	$\int_a^b \pi (\text{outer radius})^2 - (\text{inner radius})^2 \, dy$ 	$\int_a^b \pi (\text{outer radius})^2 - (\text{inner radius})^2 \, dx$ 
Shells.	$\int_a^b 2\pi x h \, dx$ 	$\int_a^b 2\pi y h \, dy.$