Math 8 - Lecture 9

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Applications of Riemann sum and integrals

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For this lecture, we want you to understand how integration can be used in different contexts. We will focus on two applications, namely probability, and the physical notion of work.

Reference (for work): James STEWART. Calculus, 8th edition. Section 5.4.

Probability density function.

- The probability an event X happens is a number between 0 and 1, denoted $P(X)$. Examples of events include winning one million dollar, waiting 2:33 at the checkout of your favorite store, having 4 kids, ...
- If all the events that can occur in a given situation $X_1, X_2, \dots, X_n$, and if two events cannot happen at the same time, then $\sum_{i=1}^{n} P(X_i) = 1$. Example: Let X_i be the event that the next time I roll a dice, I get i. Then,

$$P(X_1)+P(X_2)+P(X_3)+P(X_4)+P(X_5)+P(X_6)=1.$$

X_1 up to X_6 are the only possible events, and if I roll 2, I do not roll 4.

- If there are infinitely many events, we represent them using an integral.

For example, the probability that if I throw a ball, I throw it exactly π meters away is 0, just as the probability of throwing it exactly 10.3333... meters away. However, the total probability of throwing it within 25 meters is non-zero. This is because there are infinitely many events.

If our events are $x \in \mathbb{R}$, then $0 \leq P(x) \leq 1$, and $\int_{-\infty}^{\infty} P(x)\,dx = 1$.

Definition

The function $f(x) = P(x)$ is the density function, and $F(x) = \int_{-\infty}^{x} P(u)\,du$ is the mass function.

The mass function represents the probability of all the events "smaller than x" to happen.

Example

The lifetime, in years, of a certain class of light bulbs has density function

$$f(x) = \tfrac{1}{2} e^{-x/2}, \quad \text{if } x \geq 0.$$

1- What is the probability a given bulb will last at most two years?

This is the probability it will last exactly x years, for $0 \leq x \leq 2$.

Then, this is

$$\int_0^2 p(x)\,dx = \int_0^2 \tfrac{1}{2} e^{-x/2}\,dx$$

$$= \left(-e^{-x/2}\right)\Big|_{x=0}^{2}$$

$$= -e^{-1} + e^{0}$$

$$= 1 - e^{-1}$$

2- What is the median time a lightbulb will last?

That is the time x such that

$$\int_0^x f(u)\,du = \frac{1}{2}.$$

In this case, this is

$$\frac{1}{2} = \int_0^x \tfrac{1}{2} e^{-\frac{u}{2}}\,du = \left(-e^{-\frac{u}{2}}\right)\Big|_0^x = 1 - e^{-\frac{x}{2}}.$$

To find x, we need to solve

$$\frac{1}{2} = 1 - e^{-\frac{x}{2}}$$

$$\Rightarrow \quad \frac{1}{2} = e^{-\frac{x}{2}}$$

$$\Rightarrow \quad \ln\left(\tfrac{1}{2}\right) = -\frac{x}{2}$$

$$\Rightarrow \quad -2\ln\left(\tfrac{1}{2}\right) = x.$$

This is 1.386 years.

Example

The blue line for the bus between Hanover and Lebanon is running every 30 minutes during daytime.

If you don't know the schedule, what is the probability you wait at most x minutes? What are the mass function and the density function?

- The time you wait is between 0 and 30 minutes. It is uniformly distributed.
- The probability you wait at most x minutes is $\frac{x}{30}$, for $0 \leq x \leq 30$ (and 0 if $x \leq 0$, 1 if $x \geq 30$). This is the mass function: $F(x) = \frac{x}{30}$.
- Since $F(x) = \int_0^x P(u)\, du$, the density function is $P(x) = \frac{1}{30}$.

Definition

In physics, the work is the product of force and displacement

A force is some quantity that, when applied on an immobile object, makes the object move.

Example of forces

- Gravity
- Friction
- Spring compression
- Pulling on something

In the SI (metric) system, the work is measured in joules (J), the displacement in meters (m), the time in seconds (s), the force in newtons ($N = kg \cdot m/s^2$).

$1 J = 1 N\, m = 1\ kg \cdot m^2/s^2$

Work when the force is constant, and so is the displacement

When the force is constant, we can simply measure the work by multiplying this force by the total displacement (Just like we compute the area of a rectangle by multiplying the base and the height.)

$W = F \cdot S.$

Computing the force

Newton's second law of motion

The force applied on an object to make it move is the product of its mass and its acceleration:

$$F = m \cdot a$$

Close to the surface of the Earth, the gravity is computed using an acceleration of $g = 9.8\ m/s^2$. (You are not expected to remember it. We will tell you if that is relevant).

Example 1

How much work is done in lifting a 1.2-kg book off the floor to a desk that is 0.7 m high?

1- Compute the force

$$F = m \cdot a,$$

where the mass is 1.2 kg and the acceleration is $g = 9.8\,m/s^2$

Thus, the force is $(1.2 \times 9.8)\,N = 11.76\,N$.

2- Compute the work

$$W = F \cdot \underbrace{s}_{\text{displacement}}$$

The displacement here is 0.7 m. Hence, $W = 11.76\,N \cdot 0.7\,m$

$$= (11.76 \cdot 0.7)\,J$$
$$= 8.232\,J.$$

Work when the force or the displacement is not constant.

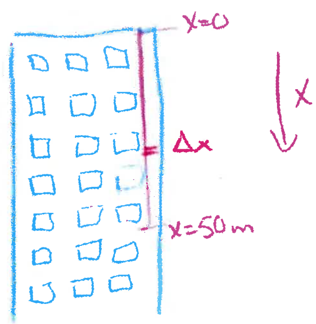
We split the displacement into a Riemann sum, and we solve the problem doing an integral.

Example

A 200-kg cable is 50 m long and hangs vertically from a tall building. How much work is required to lift the cable to the top of the building?

Here, the work is not constant along the cable, because the displacement is not the same.

We cut the cable into very thin pieces.



We divide the cable into n parts of length $\Delta x = \frac{50}{n}$. From the top the total displacement of the i-th part is x_i^*. The force is always given by the gravity, and this is $m \cdot 9.8\ m/s^2$, where the mass is $m = \frac{200}{n}$

Hence, the work done by the i-th part is

$$W_i = \underbrace{9.8 \cdot \frac{200}{n}}_{F} \underbrace{x_i^*}_{S} = 9.8 \cdot 4 \cdot \Delta x \cdot x_i^*$$

and the total work is $W = \sum_{i=1}^{n} W_i$.

While $n \to \infty$, we can transform it into an integral, with $\Delta x \to dx$ and $x_i^* \to x$.

$$W = \int_0^{50} 9.8 \cdot 4\, x\, dx = \left(\frac{39.2}{2} x^2 \right|_0^{50} = \frac{39.2 \cdot 2500}{2} = 49\,000\ J$$