Math 8 x-hour 2 Nadia Lafrenière

Problem with error bound for approximation 01/14/202

Problem

For what values of x are you sure that $|T_5(x) - \sin(x)| < 0.001$, where $T_5(x)$ is the degree-5 Maclaurin polynomial?

Solution

- Step 1: Identify Maclaurin series and $T_5(x)$.

Let $f(x) = \sin(x)$. Then

$f(0) = 0$

$f'(0) = \cos(0) = 1$

$f''(0) = -\sin(0) = 0$

$f^{(3)}(0) = -\cos(0) = -1$

$f^{(4)}(0) = \sin(0) = 0.$

$\vdots$

$$T_5(x) = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!}$$

The Maclaurin series for $\sin(x)$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

- Step 2: Compute the error. If it is an alternating series, bound it.

$$|T_5(x) - \sin(x)| = \left| x - \frac{x^3}{3!} + \frac{x^5}{5!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right|$$

Since $\sin(x)$ is equal to its Maclaurin series for all x

$$= \left| -\sum_{n=3}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right|$$

— alternating, at least when $\frac{x^{2n+1}}{(2n+1)!} < 1$, which is the case when $|x| < 2$ (and more values)

alternating series $\leq \left| \frac{x^7}{7!} \right|$

· Step 3 : Find out when that error is smaller than 0.001. (2

$$\frac{x^7}{7!} < \frac{1}{1000} \Rightarrow x < \sqrt[7]{\frac{5040}{1000}} = \sqrt[7]{5.04}$$

which is smaller than 2, since $2^7 = 128$.

· Conclusion : If $0 \leq |x| \leq \sqrt[7]{5.04} \sim 1.26$, then we are sure $|T_5(x) - \sin(x)| < 0.001$.

+ picture on Geogebra to check this is accurate.