Math 8
Winter 2020
Section 1
January 6, 2020

Please work on these problems together in small groups. Talking out loud (about the problems) is encouraged. So is comparing notes between groups.

There will generally be more problems than any one group will finish. I expect everyone to get through the first few problems. The very end of the worksheet will sometimes be optional, and sometimes covered later in class or in the reading. In either case, you can always come to office hours or tutorial to ask questions about those problems.

1. **Problem:** The tangent line approximation $T(x)$ to a function $f(x)$ near the point a is a function whose graph is a line,

$$T(x) = c_1(x - a) + c_0$$

where c_1 and c_0 are constants, that has the same value and derivative as $f(x)$ at the point a. Because we know a lot about equations for lines, we can figure out that

$$c_0 = f(a) \qquad c_1 = f'(a), \text{ so}$$

$$T(x) = f(a) + f'(a)(x - a).$$

Problem: Find the tangent line approximation to the function

$$f(x) = \ln(2x)$$

near the point .5, and use it to approximate $f(.51)$.

2. **Problem:** Use the tangent line approximation to $f(x) = \cos x$ near $x = 0$ to approximate $\cos(.1)$.

Note: $\cos(.1) \approx .995004$.

3. That wasn't very satisfying, was it? We can do a better job of approximating f near a by using a degree 2 polynomial, and matching not only the value and first derivative, but also the second derivative, at a. With a cubic we could do even better. Fortunately, thanks to calculus, we know a lot about formulas for polynomial functions, too.

   Problem: Suppose $Q(x) = c_2(x-a)^2 + c_1(x-a) + c_0$, where c_2, c_1, and c_0 are constants. Find:

$$Q(a) =$$
$$Q'(a) =$$
$$Q''(a) =$$

   Find a degree 2 polynomial

$$Q(x) = c_2x^2 + c_1x + c_0$$

   that has the same value, first derivative, and second derivative at $x = 0$ as the function $f(x) = \cos x$. Use $Q(x)$ to approximate $\cos(.1)$.

   Note: $\cos(.1) \approx .995004$. Your approximation should be much closer this time.

4. For any n, we can find a polynomial $T_n(x)$ of degree n that has the same value and first n derivatives as the function $f(x)$ at the point a. This polynomial is called the n^{th} *Taylor polynomial* for $f(x)$ at the point a (or centered at a), and it is defined by

$$T_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f^{(3)}(a)}{2\cdot 3}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{2\cdot 3\cdot 4\cdots n}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The expression $k!$, pronounced "k factorial," is defined by $k! = 1\cdot 2\cdot 3\cdot 4\cdots k$, except that for $k = 0$ we define $0! = 1$. The expression $f^{(k)}$ denotes the k^{th} derivative of f; we say $f^{(0)} = f$.

A Taylor polynomial centered at 0 is also called a Maclaurin polynomial.

Problem: Find the fifth Taylor polynomial, $T_5(x)$, for $f(x)$ at the point a, where

(a) $f(x) = \sin(x)$ and $a = 0$;

(b) $f(x) = \ln x$ and $a = 1$.

5. If we consider the function $f(x) = \sin x$ and the point $a = 0$, as we graph $T_n(x)$ for larger and larger values of n, the graphs of $T_n(x)$ look more and more like the graph of $f(x)$. That is what we would hope.

   If we do the same thing with the function $f(x) = \ln(x)$ and the point $a = 1$, the graphs of $T_n(x)$ look more and more like the graph of $f(x)$ on the interval $0 < x < 2$, but not for $x > 2$.

   If we actually want to use $T_n(x)$ as an approximation for $f(x)$, we would like to know whether it is a good approximation. We'd also like to know just how good.

   First, some notation. If $T_n(x)$ is the n^{th} Taylor polynomial for $f(x)$ at the point a, we let $R_n(x)$ denote the difference between the actual value $f(x)$ and the approximate value $T_n(x)$, called the n^{th} remainder:

$$R_n(x) = f(x) - T_n(x).$$

   The error in the approximation is the absolute value of the remainder, $|R_n(x)|$. The approximation is good if the error is small. We hope that we can make the approximation as good as we want just by making n large enough.

   Problem: Answer these questions about the Taylor polynomials of the function $f(x) = \dfrac{1}{1-x} = (1-x)^{-1}$ at the point $a = 0$.

   (a) First, a preliminary exercise.

      i. Expand and simplify the product

$$(1 + r + r^2 + \cdots + r^n)(1 - r).$$

      (There should be a lot of canceling out.)

$$(1 + r + r^2 + \cdots + r^n)(1 - r) = \underline{\hspace{6cm}}.$$

      ii. From part (i),

$$(1 + r + r^2 + \cdots + r^n) = \underline{\hspace{6cm}}.$$

(b) i. Find a formula for the k^{th} derivative of f. Recall that $f(x) = \dfrac{1}{1-x} = (1-x)^{-1}$.

$f^{(k)}(x) =$ __________________________.

ii. Find a formula for the n^{th} Taylor polynomial for $f(x)$ around the point $a = 0$. (For this problem, you may use formal summation notation $\sum$, or more informal notation using $\cdots$, whichever you prefer.)

$T_n(x) =$ __________________________.

iii. Use part (a)(ii) to rewrite this in a simpler form.

$T_n(x) =$ __________________________.

iv. Now find an expression for the n^{th} remainder.

$R_n(x) =$ __________________________.

(c) For which of the following values of x can you make the error $|R_n(x)|$ as small as you like by making n large enough?

$$x = \frac{1}{2}$$

$$x = -\frac{1}{2}$$

$$x = -1$$

$$x = 2$$

$$x = -2$$

(d) Why isn't $x = 1$ included in part (c)?

6. **Problem:** If you have finished the other problems, write out the first few terms of the 100^{th} Taylor polynomial for $f(x)$ at the point a in each of the following important examples. Make sure to include enough terms so that the pattern is clear.

   (a) $f(x) = e^x$
   $a = 0$

   (b) $f(x) = \ln(x + 1)$
   $a = 0$

   (c) $f(x) = \sin(x)$
   $a = 0$

   (d) $f(x) = \cos(x)$
   $a = 0$

   (e) What do you notice about the derivative of the 100^{th} Taylor polynomial for $f(x) = \sin(x)$ at $a = 0$?