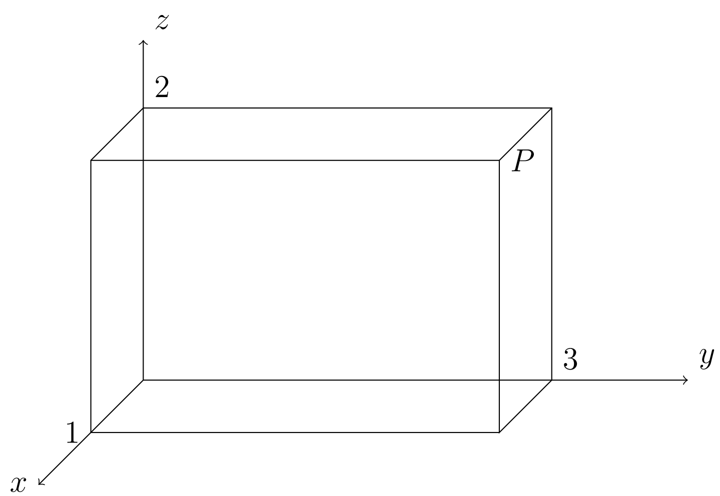
Math 8
Winter 2020
Section 1
January 29, 2020

First, some important points from the last class:

Three-dimensional coordinate system (x-axis points out of paper):



$$P = (1, 3, 2)$$

Distance from $Q = (x_1,\, y_1,\, z_1)$ to $P = (x_2,\, y_2,\, z_2)$ is

$$\boxed{|QP| = \sqrt{(x^2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$\mathbb{R}^3$ denotes 3-dimensional space, or the set of all triples (a, b, c) of real numbers.

Sphere of radius r with center (a, b, c):

$$\boxed{(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2}$$

Projections:

The projection of a point P on a plane, or on a line, is the point on that plane or line you reach from P by proceeding perpendicularly to the plane or line.

Example: If $P = (1, 2, 3)$, the projection of P on the xy-plane is $(1, 2, 0)$ and the projection of P on the x-axis is $(1, 0, 0)$.

The projection of a shape S on a plane or line consists of the projections of all the points in S on that plane or line.

You can think of the projection of S on a plane as the shadow of S cast on that plane by light shining perpendicularly to the plane.

Example: If S is the sphere around $(0, 0, 3)$ with radius 2, the projection of S on the xy-plane is the disc centered at the origin with radius 2, and the projection of S on the z-axis is the interval $1 \leq z \leq 4$.

Note: Don't confuse a projection with an intersection or cross-section. If S is the sphere $x^2 + y^2 + z^2 = 1$ around the origin with radius 1, the *projection* of S on the xy-plane is the disc $x^2 + y^2 \leq 1$. However, the *intersection* of S with the xy-plane, or the *cross-section* of S at $z = 0$, is the circle $x^2 + y^2 = 1$.

Vectors:

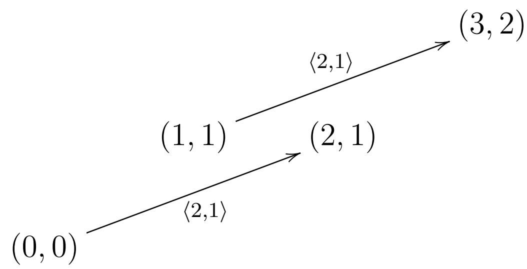
We will think about vectors in 3 ways: algebra, geometry, and applications.

Algebra: A vector in $\mathbb{R}^n$ is an n-tuple of real numbers $\langle a_1, a_2, \ldots, a_n \rangle$. (We often use angle brackets to distinguish vectors from points.)

For us, usually $n = 2$ or $n = 3$.

Geometry: A vector $\langle a_1, a_2, \ldots, a_n \rangle$ can be drawn as an arrow from the point $(0, 0, \ldots, 0)$ to the point $(a_1, a_2, \ldots, a_n)$ OR it can be drawn as an arrow of the same length and direction starting at any other point.

Two arrows with the same length and direction are two pictures of the same vector.



Going along an arrow representing $\langle a_1, a_2, a_3 \rangle$ you go a_1 units in the x-direction, a_2 units in the y-direction, and a_3 units in the z-direction.

Applications: Vectors are used to model anything that is given by its magnitude (represented by the length of the vector) and direction (represented by the direction of the vector).

Examples:

1. The position vector of the point P with coordinates (a, b, c) is the vector $\langle a, b, c \rangle$ from the origin to P.

   Note: The position vector of P is a vector, so it can be drawn starting at any point. When it is drawn starting at the origin, it ends at P.

2. If an object moves from point P to point Q, its *displacement* is represented by the vector that can be drawn with tail (start) at P and head (point) at Q.

   Displacement gives direction of motion and distance, but not starting position. So an object that moves from $(0, 0, 0)$ to $(1, 1, 1)$ and an object that moves from $(1, 2, 3)$ to $(2, 3, 4)$ have the same displacement, $\langle 1, 1, 1 \rangle$.

3. Velocity is represented by a vector. Its direction is the direction of motion, and its length gives the speed.

4. Force is represented by a vector.

5. Acceleration is represented by a vector.

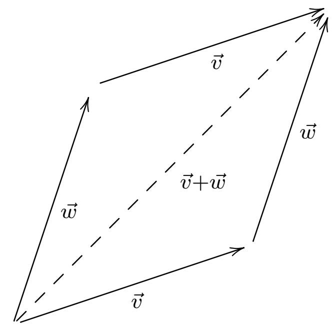
   Note: An object can have constant speed but nonzero acceleration, if its direction of motion is changing. We will be able to analyze acceleration later.

Vector addition: (Add two vectors, result is a vector.)

Algebra: Add vectors coordinatewise.

$$\langle a_1, b_1, c_1 \rangle + \langle a_2, b_2, c_2 \rangle = \langle a_1 + a_2, b_1 + b_2, c_1 + c_2 \rangle$$

Geometry: Parallelogram law.



Applications:
Net displacement, resultant force, net velocity.

Scalar multiplication: (Multiply a vector by a real number, result is a vector.)

Algebra: Multiply coordinates by the real number (scalar).

$$k \langle a_1, b_1, c_1 \rangle = \langle ka_1, kb_1, kc_1 \rangle$$

Geometry: The length of $k\vec{v}$ is $|k|$ times the length of $\vec{v}$. The direction of $k\vec{v}$ is the same as the direction of $\vec{v}$ if $k > 0$, and the opposite if $k < 0$.

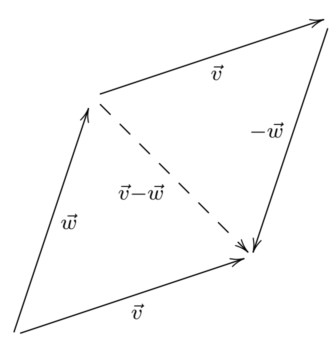
Subtraction: (Subtract one vector from another, result is a vector.)

Algebra:

$$-\vec{v} = (-1)\vec{v} \quad \text{and} \quad \vec{v} - \vec{w} = \vec{v} + (-\vec{w})$$

$$\langle a_1, b_1, c_1 \rangle - \langle a_2, b_2, c_2 \rangle = \langle a_1 - a_2, b_1 - b_2, c_1 - c_2 \rangle$$

Geometry: A different parallelogram law.



Magnitude (norm) of vector: (A real number, or scalar.)

Geometry: $|\vec{v}|$ is the length of $\vec{v}$.
Algebra: Use the distance formula:

$$|\langle a, b, c\rangle| = \sqrt{a^2 + b^2 + c^2}$$

Applications: If $\vec{v}$ represents displacement, $|\vec{v}|$ represents distance; if $\vec{v}$ represents velocity, $|\vec{v}|$ represents speed; if $\vec{v}$ represents force, $|\vec{v}|$ represents the magnitude of the force.

Zero vector:

The zero vector is the vector $\vec{0} = \langle 0, 0, 0\rangle$ whose coordinates are all 0.
The zero vector has length 0 and has no direction.

Unit vector:

A unit vector is any vector whose magnitude is 1.
We often use unit vectors to specify direction.

Parallel vectors:

Geometry: Two nonzero vectors are parallel if they have either the same direction or opposite directions.
Algebra: Two nonzero vectors are parallel if one is a scalar multiple of the other.

Notation: Sometimes vectors are written with an arrow on top, $\vec{v}$. Sometimes, instead, they are written in boldface, $\mathbf{v}$.
Sometimes the norm of a vector $\vec{v}$ is written $||\vec{v}||$ instead of $|\vec{v}|$.
Some texts that always use boldface for vectors, use v for the norm of $\mathbf{v}$.

Example: Find a unit vector whose direction is the direction of motion of an object moving in a straight line from $(0, 1, 0)$ to $(3, 5, 12)$.

Theorem: If $\vec{v}$ is a nonzero vector, the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{1}{|\vec{v}|}\,\vec{v}.$$

Standard basis for $\mathbb{R}^2$:

$$\{\hat{i}, \hat{j}\}$$

$$\hat{i} = \langle 1, 0 \rangle$$

$$\hat{j} = \langle 0, 1 \rangle$$

$$\langle a, b \rangle = a\hat{i} + b\hat{j}$$

$$\vec{e}_1 = \hat{i} \qquad \vec{e}_2 = \hat{j}$$

Standard basis for $\mathbb{R}^3$:

$$\{\hat{i}, \hat{j}, , \hat{k}\}$$

$$\hat{i} = \langle 1, 0, 0 \rangle$$

$$\hat{j} = \langle 0, 1, 0 \rangle$$

$$\hat{k} = \langle 0, 0, 1 \rangle$$

$$\langle a, b, c \rangle = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{e}_1 = \hat{i} \qquad \vec{e}_2 = \hat{j} \qquad \vec{e}_3 = \hat{k}$$

Standard basis for $\mathbb{R}^n$:

$$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$$

$$\vec{e}_i = \left\langle 0, \ldots, 0, \underbrace{1}_{i^{th} \text{ position}}, 0 \ldots, 0 \right\rangle$$

Notation: It is common, particularly in physics and engineering, to write the vector $\langle a, b, c \rangle$ as $a\hat{i} + b\hat{j} + c\hat{k}$, or as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Example: A weight hangs from 2 ropes. The weight is 3 meters below a point on the ceiling, one rope is anchored 3 meters to the left of that point, the other anchored $3\sqrt{3}$ meters to the right of that point. The magnitude of the tension in the first rope (the tension is the force exerted by the first rope) is measured at 3 newtons. Find the mass of the weight.

Physics you need to use:

The force on the weight exerted by each rope (the tension) has direction from the weight to the point at which the rope is anchored.

In addition to the forces exerted by the two ropes, there is a third force acting on the weight, the force of gravity, which is directed straight down and has magnitude mg newtons, where m is the mass of the weight in kilograms and g is the acceleration of gravity near the earth's surface, approximately $9.8\frac{\mathrm{m}}{\mathrm{s}^2}$.

Because the weight is hanging still, the net force on the weight is zero.

Exercise: We set up our axes with the z-axis vertical, the x-axis pointing east, and the y-axis pointing north. An airplane's velocity relative to the air around it is $\langle 150, 200, 5 \rangle$, with units of miles and hours. If the wind is blowing from the northeast at 15 miles per hour (and parallel to the ground), what is the airplane's velocity relative to the ground?

Exercise: This example uses SI units of meters and seconds. Find a vector that represents the velocity of an object moving in a straight line from the origin toward the point $(15, 20, 60)$ (where coordinates are in meters) at a speed of .5 meters per second. (Suggestion: First find any vector in the direction of motion. Then find the velocity vector.)

If the object starts from the origin at time $t = 0$, where will it be at time $t = 6$? (Suggestion: Find its displacement, by finding its direction of motion and the distance it travels.)

Exercise: Express the vector $\langle 5, 0 \rangle$ in terms of $\langle 1, 2 \rangle$ and $\langle -2, 1 \rangle$ by solving for the scalars A and B in

$$\langle 5, 0 \rangle = A \langle 1, 2 \rangle + B \langle -2, 1 \rangle$$

Can every vector in $\mathbb{R}^2$ be expressed in terms of $\langle 1, 2 \rangle$ and $\langle -2, 1 \rangle$ in this way? (Think geometrically: Can every vector be obtained by adding a vector parallel to $\langle 1, 2 \rangle$ and a vector parallel to $\langle -2, 1 \rangle$?)

Challenge Problem: If an object starts at a point with position vector $\vec{r}_0$ and moves with velocity $\vec{v}$ for t units of time, what is the position vector of its final location?

Cultural enrichment:

Definition: A *vector space* (over $\mathbb{R}$) consists of objects that can be added to each other and multiplied by real numbers (scalars). There is an object called $\mathbf{0}$, and for each object $\mathbf{x}$ there is an associated object $-\mathbf{x}$. Addition and scalar multiplication follow these rules ($\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{0}$ are vectors, a, b, 1 are scalars):

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

$$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$$

$$\mathbf{x} + \mathbf{0} = \mathbf{x}$$

$$\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$$

$$1\mathbf{x} = \mathbf{x}$$

$$a(b\mathbf{x}) = (ab)\mathbf{x}$$

$$a(\mathbf{x} + \mathbf{y}) = (a\mathbf{x}) + (\mathbf{y})$$

$$(a + b)\mathbf{x} = (a\mathbf{x}) + (b\mathbf{x})$$

Examples of vector spaces:

$\mathbb{R}^3$
$\mathbb{R}^n$
The set of all 2×3 matrices with real number entries.
The set of all quadratic polynomials with real number coefficients.
The set of all polynomials with real number coefficients.
The set of all continuous functions from $[0, 1]$ to $[0, 1]$.
The complex numbers $\mathbb{C}$.

$\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$ spans a vector space if every $\mathbf{x}$ in the vector space can be expressed as

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n$$

where $a_1, a_2, \ldots, a_n$ are real numbers.

$\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$ is a *basis* for a vector space if it spans the vector space, but no smaller subset does.

A vector space has dimension n if it has a basis of size n.

Examples:

$\mathbb{R}^3$ has dimension 3 (one basis is $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$); $\mathbb{R}^n$ has dimension n.

The set of quadratic polynomials has dimension 3. A basis is $\{1, x, x^2\}$.

The set of 2×3 matrices has dimension 6. The matrices with a 1 in one entry and 0 in all other entries comprise a basis.

$\mathbb{C}$ has dimension 2. A basis is $\{1, i\}$.

The other examples above have infinite dimension.