Math 8
Winter 2020
Section 1
February 3, 2020

First, some important points from the last class:

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}|\cos(\theta),$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

$$\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The component of $\vec{F}$ on (or along, or in the direction of) $\vec{d}$ is

$$comp_{\vec{d}}(\vec{F}) = |\vec{F}|\cos(\theta) = \boxed{\frac{\vec{F} \cdot \vec{d}}{|\vec{d}|}}$$

and the projection of $\vec{F}$ onto $\vec{d}$ is

$$proj_{\vec{d}}(\vec{F}) = \underbrace{\left(|\vec{F}|\cos(\theta)\right)}_{\text{component}} \underbrace{\left(\frac{1}{|\vec{d}|}\,\vec{d}\right)}_{\text{unit vector}} = \boxed{\left(\frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\right)\vec{d}}$$

The work done by force $\vec{F}$ on an object moving in a straight line with displacement $\vec{d}$ is

$$\boxed{W = \vec{F} \cdot \vec{d}}$$

"Work equals force dot displacement."

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

$$\vec{v} \cdot (\vec{w} + \vec{u}) = (\vec{v} \cdot \vec{w}) + (\vec{v} \cdot \vec{u})$$

$$\vec{v} \cdot (\vec{w} - \vec{u}) = (\vec{v} \cdot \vec{w}) - (\vec{v} \cdot \vec{u})$$

$$(t\vec{v}) \cdot \vec{w} = t(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (t\vec{w})$$

$$\vec{0} \cdot \vec{v} = 0$$

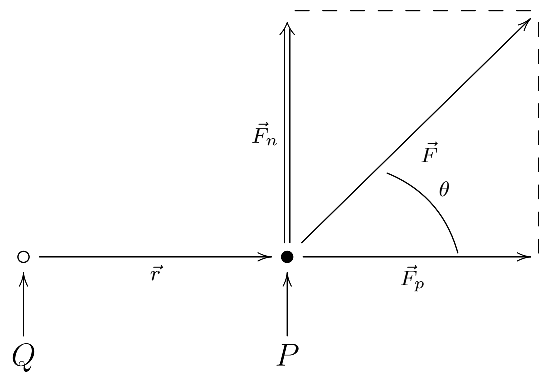
$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

Theorem (the triangle inequality):

$$|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}|$$

Theorem (the Cauchy-Schwartz inequality):

$$|\vec{v} \cdot \vec{w}| \leq |\vec{v}|\,|\vec{w}|$$



An object • (at point P) is attached by a rigid rod to a fixed point ∘ (point Q), but is free to rotate around that point in any direction. The vector $\vec{r}$ goes to the object from the fixed point around which it may rotate. A force $\vec{F}$ acts on the object.

The torque vector $\vec{\tau}$ represents the tendency of the object to rotate around the fixed point, caused by the force $\vec{F}$.

If we decompose $\vec{F}$ into two component forces, $\vec{F}_p$ parallel to $\vec{r}$ and $\vec{F}_n$ normal to $\vec{r}$, only $\vec{F}_n$ imparts torque.

The magnitude of the torque depends both on the force and on the distance from the fixed point (think levers, or seesaws), and is

$$|\tau| = |\vec{r}|\,|\vec{F}_n| = |\vec{r}|\,|\vec{F}|\sin(\theta).$$

The direction of τ gives the direction of the axis around which the object rotates.

> Repetition for emphasis: The direction of τ gives the **direction of the axis** around which the object rotates. It does *not* give the direction in which the object moves.
>
> This is the way we represent rotational motion. As the earth rotates around its axis, different points on its surface are moving in different directions, but the axis of rotation is the same for the entire globe. So torque, rotational (angular) velocity, etc. are represented by vectors pointing along the axis of rotation.

In this picture, since $\vec{r}$ and $\vec{F}$ are both in the plane of the paper, the axis of rotation is perpendicular to the paper, so $\vec{\tau}$ points in a direction perpendicular to the paper — either out or in. By convention, since the rotation is counterclockwise as we look at the paper, the torque vector τ points out of the paper toward us.

> **Right-hand rule:** Suppose a force $\vec{F}$ acts on an object at point P, causing a rotational tendency around point Q, and $\vec{r}$ is the vector from the fixed point Q to the object at P, inducing a torque $\vec{\tau}$ around the point Q. If you point the thumb of your right hand in the direction of τ and curl your fingers, they will be pointing around $\vec{\tau}$ from the vector $\vec{r}$ toward the vector $\vec{F}$.

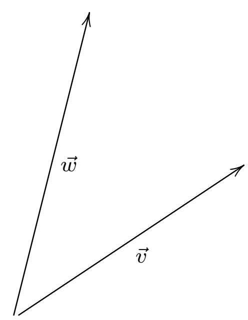
Definition: The cross product, or vector product, of vectors $\vec{r}$ and $\vec{F}$ is the vector $\vec{r} \times \vec{F}$ with the following properties:

1. $|\vec{r} \times \vec{F}| = |\vec{r}|\,|\vec{F}|\sin(\theta)$ where θ is the angle between $\vec{r}$ and $\vec{F}$.

2. $\vec{r} \times \vec{F}$ is perpendicular to both $\vec{r}$ and $\vec{F}$.

3. $\vec{r}$, $\vec{F}$ and $\vec{r} \times \vec{F}$ are oriented according to the right-hand rule: If all three vectors are drawn from the same point, and you are looking down from the top of $\vec{r} \times \vec{F}$, rotating from $\vec{r}$ around to $\vec{F}$ appears as a counterclockwise rotation.

Note: The cross product is defined only in $\mathbb{R}^3$.



$\vec{v} \times \vec{w}$ points out of the paper. $\vec{w} \times \vec{v}$ points into the paper.

Two other ways to remember the right-hand rule:

Hold your arms out parallel to the ground and pointing slightly forwards, at an angle to each other. If your right hand points in the direction of the first vector $\vec{v}$ and your left hand points in the direction of the second vector $\vec{w}$, then your head points in the direction of the vector $\vec{v} \times \vec{w}$. (Provided, of course, that you haven't crossed your arms.)

The vectors $\vec{v}$, $\vec{w}$, and $\vec{v} \times \vec{w}$, in that order, are oriented in the same way as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, in that order. And, in fact, $\hat{\imath} \times \hat{\jmath} = \hat{k}$.

Example: Find the vector $\hat{k} \times \hat{\jmath}$.

Algebra of the cross product.

The determinant of a matrix will help us compute cross products without getting too mixed up.

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} =$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{***} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

*** is the determinant of the matrix left when you cross out the row and column of a_1.
Notice the alternating $+$ and $-$ signs.

The matrix determinant has many useful applications. We're going to use it in the formula for cross product:

$$\langle v_1, v_2, v_3 \rangle \times \langle w_1, w_2, w_3 \rangle =$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \hat{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \hat{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \hat{k} =$$

$$\langle v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1 \rangle .$$

Example:

$$\hat{k} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \hat{k} =$$

$$((0)(0) - (1)(1))\hat{i} - ((0)(0) - (0)(1))\hat{j} + ((0)(1) - (0)(0))\hat{k} = -\hat{i}$$

Example: Compute $\langle 1, 2, 1 \rangle \times \langle 1, 0, -1 \rangle$. Use the dot product to check that the cross product is orthogonal to both factors.

Theorem: $|\vec{v} \times \vec{w}|$ is the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$.

Theorem:

$$\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$$

$$t(\vec{v} \times \vec{w}) = t\vec{v} \times \vec{w} = \vec{v} \times t\vec{w}$$

$$\vec{v} \times (\vec{w} + \vec{u}) = (\vec{v} \times \vec{w}) + (\vec{v} \times \vec{u})$$

$$\vec{v} \times (\vec{w} \times \vec{u}) = (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}$$

Definition: The *triple product* of $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$, and $\vec{u} = \langle u_1, u_2, u_3 \rangle$, in that order, is

$$\vec{v} \cdot (\vec{w} \times \vec{u}) = (\vec{v} \times \vec{w}) \cdot \vec{u} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

Theorem: The absolute value of the triple product of $\vec{v}$, $\vec{w}$, and $\vec{u}$ is the volume of the parallelepiped with edges $\vec{v}$, $\vec{w}$, and $\vec{u}$.

The triple product is positive if $\vec{v}$, $\vec{w}$, and $\vec{u}$ are oriented according to the right hand rule in the same way as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ (or as $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$). It is negative if they have the opposite orientation.

(This is related to some of the ways in which determinants are useful.)

Example: What does it mean geometrically if $\vec{v} \times \vec{w} = \vec{0}$?

If $|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}|$?

If $\vec{v} \cdot (\vec{w} \times \vec{u}) = 0$?

If $|\vec{v} \cdot (\vec{w} \times \vec{u})| = |\vec{v}|\,|\vec{w}|\,|\vec{u}|$?

Is it always true that $|\vec{v} \cdot (\vec{w} \times \vec{u})| \leq |\vec{v}|\,|\vec{w}|\,|\vec{u}|$? Why or why not?

A plane contains the triangle with corners $(1, 1, 1)$, $(1, 2, 3)$, and $(2, 2, -1)$.
Find two vectors parallel to the plane but not parallel to each other.

Find a vector perpendicular to the plane. (Hint: Use the cross product.)

(We will see next time that if you know a point on a plane and a vector perpendicular to the plane, you can write down the equation of the plane, so this is a useful thing to be able to do.)

Find the volume of the parallelepiped with edges
$\langle 1, 2, 1\rangle, \langle -1, 0, 1\rangle, \langle 1, 1, 2\rangle$.

Use the algebraic rules for dot products and cross products (for example, the distributive law $\vec{v} \times (\vec{w} + \vec{u}) = (\vec{v} \times \vec{w}) + (\vec{v} \times \vec{u})$) to show that the triple product of $\vec{v}$, $\vec{w}$, and $s\vec{v} + t\vec{w}$ is always zero, for any vectors $\vec{v}$ and $\vec{w}$ and any scalars s and t.

Now use geometric reasoning to show the same thing.

Example (from last time):

1. What does it mean geometrically if

$$(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = 0?$$

It means that $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$ are perpendicular to each other.

By the parallelogram laws for addition and subtraction of vectors, this means the diagonals of the parallelogram whose edges are $\vec{v}$ and $\vec{w}$ are perpendicular to each other.

2. Use the basic facts about dot products to show that

$$(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 - |\vec{w}|^2$$

First use the distributive laws for the dot product:

$$(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot (\vec{v} - \vec{w}) + \vec{w} \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v} - \vec{w} \cdot \vec{w}$$

Now use the commutative law for the dot product:

$$\vec{v} \cdot \vec{v} \underline{- \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v}} - \vec{w} \cdot \vec{w} = \vec{v} \cdot \vec{v} \underline{- \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{w}} - \vec{w} \cdot \vec{w} = \vec{v} \cdot \vec{v} - \vec{w} \cdot \vec{w}$$

Finally, use the fact that $\vec{a} \cdot \vec{a} = |\vec{a}|^2$:

$$\vec{v} \cdot \vec{v} - \vec{w} \cdot \vec{w} = |\vec{v}|^2 - |\vec{w}|^2$$

3. Use (1) and (2) to derive a theorem about the diagonals of a parallelogram.

Theorem: The diagonals of a parallelogram are perpendicular if and only if the edges of the parallelogram have the same length.

Proof: Let $\vec{v}$ and $\vec{w}$ be the edges of the parallelogram. By (1), the diagonals are perpendicular if and only if $(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = 0$. By (2) this is true if and only if $|\vec{v}|^2 - |\vec{w}|^2 = 0$; that is, if and only if $|\vec{v}|^2 = |\vec{w}|^2$. Since the norm of a vector is never negative, this happens if and only if $|\vec{v}| = |\vec{w}|$; that is, if and only if the edges of the parallelogram have the same length.