Math 8
Winter 2020
Section 1
February 5, 2020

First, some important points from the last class:

Definition: The cross product, or vector product, of vectors $\vec{r}$ and $\vec{F}$ is the vector $\vec{r} \times \vec{F}$ with the following properties:

1. $|\vec{r} \times \vec{F}| = |\vec{r}|\,|\vec{F}|\sin(\theta)$ where θ is the angle between $\vec{r}$ and $\vec{F}$.
2. $\vec{r} \times \vec{F}$ is perpendicular to both $\vec{r}$ and $\vec{F}$.
3. $\vec{r}$, $\vec{F}$ and $\vec{r} \times \vec{F}$ are oriented according to the right-hand rule: If all three vectors are drawn from the same point, and you are looking down from the top (the pointed end) of $\vec{r} \times \vec{F}$, rotating from $\vec{r}$ around to $\vec{F}$ appears as a counterclockwise rotation.

If $\vec{r}$ goes from Q to P, and a force $\vec{F}$ is applied at P, the induced torque around the fixed point Q is $\vec{r} \times \vec{F}$.

$|\vec{r} \times \vec{F}|$ is the area of the parallelogram with edges $\vec{r}$ and $\vec{F}$.

$$\langle v_1, v_2, v_3\rangle \times \langle w_1, w_2, w_3\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Theorem:

$$\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$$
$$t(\vec{v} \times \vec{w}) = t\vec{v} \times \vec{w} = \vec{v} \times t\vec{w}$$
$$\vec{v} \times (\vec{w} + \vec{u}) = (\vec{v} \times \vec{w}) + (\vec{v} \times \vec{u})$$
$$\vec{v} \times (\vec{w} \times \vec{u}) = (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}$$

Warning: The cross product is *not* commutative and *not* associative.

Definition: The *triple product* of $\vec{v} = \langle v_1, v_2, v_3\rangle$, $\vec{w} = \langle w_1, w_2, w_3\rangle$, and $\vec{u} = \langle u_1, u_2, u_3\rangle$, in that order, is

$$\vec{v} \cdot (\vec{w} \times \vec{u}) = (\vec{v} \times \vec{w}) \cdot \vec{u} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

Theorem: The absolute value of the triple product of $\vec{v}$, $\vec{w}$, and $\vec{u}$ is the volume of the parallelepiped with edges $\vec{v}$, $\vec{w}$, and $\vec{u}$.

The triple product is positive if $\vec{v}$, $\vec{w}$, and $\vec{u}$ are oriented according to the right hand rule in the same way as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. It is negative otherwise.

Preliminary Homework Assignment

1. In a previous homework assignment, you showed the following:

   > Suppose an object starts at point (a, b, c) and moves with constant velocity $\vec{v} = \langle x_v,\, y_v,\, z_v \rangle$ for t seconds.
   > Then its final position is $(a + x_v t,\; b + y_v t,\; c + z_v t)$.

   What geometric object does the set of all points of the form $(a + x_v t,\; b + y_v t,\; c + z_v t)$ describe? (Be as specific as you can. For example, if the object were a sphere, a complete answer would not only say that, but also identify the center and radius.)

   A line through the point (a, b, c) in the direction of the vector $\vec{v} = \langle x_v,\, y_v,\, z_v \rangle$. Note we can also write

$$\underbrace{\langle x, y, z \rangle}_{\text{general point on line}} = \underbrace{\langle a, b, c \rangle}_{\text{given point on line}} + t \underbrace{\langle x_v, y_v, z_v \rangle}_{\text{vector parallel to line}} .$$

Definition: A *vector parametric equation* for the line parallel to vector $\vec{v} = \langle x_v, y_v, z_v \rangle$ passing through the point (x_0, y_0, z_0) with position vector $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ is

$$\vec{r} = \vec{r}_0 + t\vec{v}.$$

Here $\vec{r}$ is a variable vector, $\vec{r} = \langle x, y, z \rangle$, so this is

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle x_v, y_v, z_v \rangle .$$

Scalar parametric equations for this line are

$$x = x_0 + tx_v \qquad y = y_0 + ty_v \qquad z = z_0 + tz_v.$$

We can also find *symmetric scalar equations* by solving for t in the scalar parametric equations and setting the results equal (to eliminate t):

$$\frac{x - x_0}{x_v} = \frac{y - y_0}{y_v} = \frac{z - z_0}{z_v}.$$

Note that symmetric scalar equations will look a little different if one of the coordinates of v is zero. For example, if $z_v = 0$, the third scalar parametric equation is $z = z_0$, in which t is already eliminated. In this case our symmetric scalar equations become

$$\frac{x - x_0}{x_v} = \frac{y - y_0}{y_v} \quad \text{and} \quad z = z_0.$$

Example: Find a vector parametric equation for the line through the points $(1, 1, 1)$ and $(2, 3, 4)$.

Example: Find scalar parametric equations for the line through the origin parallel to this line.

Example Determine whether the lines with vector equations $\vec{r} = \langle 1, 1, 1\rangle + t\langle 2, 3, 4\rangle$ and $\vec{r} = \langle 4, 3, 5\rangle + t\langle 1, -1, 0\rangle$ intersect.

2. (a) What does the equation

$$\langle 1, 2, -1 \rangle \cdot \langle x, y, z \rangle = 0$$

say about the position vector of the point (x, y, z)?

It is normal (orthogonal, perpendicular) to the vector $\langle 1, 2, -1 \rangle$.

(b) What geometric object does the equation

$$x + 2y - z = 0$$

describe?

A plane containing the origin and perpendicular to the vector $\langle 1, 2, -1 \rangle$.

Definition: A *vector equation* for the plane perpendicular to the vector $\vec{n} = \langle a, b, c \rangle$ containing the point (x_0, y_0, z_0) with position vector $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

We can turn this into a scalar equation (a linear equation), sometimes called an implicit equation by WeBWorK, as follows:

$$\langle a, b, c \rangle \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Note: From a linear equation $ax + by + cz = d$ for a plane, you can read off a normal vector $\vec{n} = \langle a, b, c \rangle$.

Example: Find an equation for the plane containing the point $(1, 2, 5)$ parallel to the plane with equation

$$x - 2y + z = 5.$$

Example: Find a linear equation for the plane containing the points $(1, 1, 1)$, $(1, 2, 3)$, and $(-1, -1, 1)$.

Example: Find an equation for the line through the origin normal to the plane with equation

$$x - y + z = 4.$$

Find the point where this line intersects the plane.

Definition: A vector parametric equation for the plane containing the point with position vector $\vec{r}_0$ and parallel to both vectors $\vec{v}$ and $\vec{w}$ (which are not parallel to each other) is

$$\vec{r} = \vec{r}_0 + t\vec{v} + s\vec{w}.$$

Question: Find a normal vector to this plane.

$$\vec{v} \times \vec{w}$$

Definition: Planes are called parallel if they have parallel normal vectors.

The angle between planes with normal vectors $\vec{n}_1$ and $\vec{n}_2$ is the angle between $\vec{n}_1$ and $\vec{n}_2$ if this angle is between 0 and $\frac{\pi}{2}$, and the angle between $\vec{n}_1$ and $-\vec{n}_2$ otherwise.

Example: Find the distance from the point $(1,2,3)$ to the plane with equation

$$x + y + z = 3.$$

Method 1: Take a vector $\vec{v}$ from $(1,2,3)$ to any point on the plane. See the distance from the point to the plane is $|comp_{\vec{n}}(\vec{v})|$, where $\vec{n}$ is a normal vector to the plane.

Method 2: Find a line through $(1, 2, 3)$ normal to the plane. Find the point of intersection of the line and plane; that point is the point on the plane closest to $(1, 2, 3)$. Take the distance between that point and $(1, 2, 3)$.

Exercise: Find a linear equation for the plane through the origin that is parallel to both the lines $\vec{r} = \langle -1, -2, 1\rangle + t\,\langle 1, -1, 0\rangle$ and $\vec{r} = \langle 1, 0, 1\rangle + t\,\langle 1, 1, -1\rangle$.

Exercise: Find the distance between the parallel planes

$$x + 2y - z = 4$$

$$2x + 4y - 2z = 4.$$

Exercise: Does the line through the points $(7, 9, 3)$ and $(-2, -3, 0)$ intersect the line through the points $(2, 2, 3)$ and $(0, 0, -1)$?

Method 1: Write vector parametric equations of the lines in the form $\langle x, y, z\rangle = \vec{r}_0 + t\vec{v}_0$ and $\langle x, y, z\rangle = \vec{r}_1 + s\vec{v}_1$. See whether you can solve for s and t in the equation $\vec{r}_0 + t\vec{v}_0 = \vec{r}_1 + s\vec{v}_1$. (This will also find the point of intersection, if there is one. The other two methods will not.)

NOTE: It is important *not* to use the same variable t in the equations for both lines, and try to solve $\vec{r}_0 + t\vec{v}_0 = \vec{r}_1 + t\vec{v}_1$. This would correspond to finding whether two particles moving along these lines, with motion described by these equations, collide at some time t. We want to find whether their paths cross (at some point they may reach at different times s and t).

Method 2: We can see the lines are not parallel (How?), so they intersect exactly in case there is a plane containing both lines. Check whether the four points are on the same plane. To do this: Look at the three vectors from any one of the points to each of the three other points. Use the triple product to check whether those vectors are coplanar.

Method 3: Find a vector $\vec{n}$ normal to both lines. A plane with normal vector $\vec{n}$ will be parallel to both lines. Find a plane with normal vector $\vec{n}$ containing the first line (by making sure it contains some point on the line). If the lines intersect, the plane will contain the second line, and if not, the plane will not intersect the second line at all. Check any point on the second line to see whether it satisfies the equation for the plane.

Exercise: Find the distance between the skew lines (lines that are not parallel but do not meet) $\vec{r} = \langle -1, -2, 1 \rangle + t \langle 1, -1, 0 \rangle$ and $\vec{r} = \langle 1, 0, 1 \rangle + t \langle 1, 1, -1 \rangle$.

Method 1: Find a vector $\vec{n}$ that is perpendicular to both lines. Then take any points P on the first line and Q on the second line, and take the absolute value of the component of $\overrightarrow{PQ}$ in the direction of the vector $\vec{n}$.

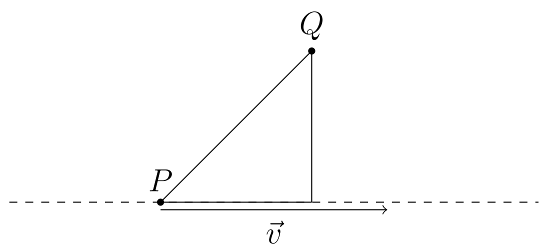
Explain why this works. (Hint at one possible explanation: Think of writing $\overrightarrow{PQ}$ as the sum of three vectors, one parallel to the first line, one perpendicular to both lines, and one parallel to the second line. Draw a picture, including both lines and both points, with these vectors positioned head-to-tail making a path starting at P and ending at Q. Explain why the length of the second leg of the path is the distance between the lines.)

Method 2: Can you think of another method?

Do not try to memorize the various formulas for the distances between a point and a line, between a point and a plane, between two lines, between a line and a plane, between two planes.

Exercise: Think of at least two methods to find the distance between a point and a line (in three dimensions).

This picture may suggest one method. We want to find the distance between the point Q and the line through point P in the direction of vector $\vec{v}$.



Some solutions to problems from last time:

1. A plane contains the triangle with corners $(1,1,1)$, $(1,2,3)$, and $(2,2,-1)$.

a. Find two vectors parallel to the plane but not parallel to each other.

Solution: We can take the displacement vectors from $(1,1,1)$ to $(1,2,3)$ and from $(1,1,1)$ to $(2,2,-1)$. We get these by subtracting the coordinates of the starting point from the coordinates of the ending point, so they are $\langle 0,1,2\rangle$ and $\langle 1,1,-2\rangle$.

b. Find a vector perpendicular to the plane. (Hint: Use the cross product.)

Solution: Take the cross product of two vectors parallel to the plane,
$\langle 0,1,2\rangle \times \langle 1,1,-2\rangle = \langle -4,2,-1\rangle$.

2. Find the volume of the parallelepiped with edges
$\langle 1,2,1\rangle, \langle -1,0,1\rangle, \langle 1,1,2\rangle$.

Solution: Take the absolute value of the triple product,
$|\langle 1,2,1\rangle \cdot (\langle -1,0,1\rangle \times \langle 1,1,2\rangle)| = |\langle 1,2,1\rangle \cdot \langle -1,3,-1\rangle| = 4$, or $\left| det \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \right| = 4.$