Math 8
Winter 2020
Section 1
February 7, 2020

First, some important points from the last class:

Definition: A vector parametric equation for the line parallel to vector $\vec{v} = \langle x_v, y_v, z_v \rangle$ passing through the point (x_0, y_0, z_0) with position vector $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is

$$\vec{r} = \vec{r_0} + t\vec{v}, \text{ or}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle x_v, y_v, z_v \rangle .$$

Scalar parametric equations for this line are

$$x = x_0 + tx_v \quad y = y_0 + ty_v \quad z = z_0 + tz_v.$$

Definition: A vector equation for the plane perpendicular to the vector $\vec{n} = \langle a, b, c \rangle$ containing the point (x_0, y_0, z_0) with position vector $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

The scalar (linear) equation is:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Note: From a linear equation $ax + by + cz = d$ for a plane, you can read off the normal vector $\vec{n} = \langle a, b, c \rangle$.

Definition: A vector parametric equation for the plane containing the point with position vector $\vec{r_0}$ and parallel to both vectors $\vec{v}$ and $\vec{w}$ (which are not parallel to each other) is

$$\vec{r} = \vec{r_0} + t\vec{v} + s\vec{w}.$$

Definition: Planes are called parallel if they have parallel normal vectors.

The angle between two planes is the acute angle between their normal vectors.

Preliminary Homework Assignment

In a previous homework assignment, you showed the following:

> Suppose an object starts at point (a, b, c) and moves with constant velocity $\vec{v} = \langle v_x,\, v_y,\, v_z \rangle$ for t seconds.
>
> Then its final position is $(a + v_x t,\, b + v_y t,\, c + v_z t)$.

We can express this by a function whose domain is the real number line $\mathbb{R}$ and whose range lies in the three-dimensional space $\mathbb{R}^3$,

$$\vec{f}(t) = (a + v_x t,\, b + v_y t,\, c + v_z t),$$

where t represents time, with $t = 0$ being the starting time, and $\vec{f}(t)$ is the object's position vector at time t.

Another object is traveling clockwise around the unit circle $x^2 + y^2 = 1$ in the plane $\mathbb{R}^2$. At time $t = 0$ it is at the point $(1, 0)$, and it travels at constant speed, making one complete trip around the circle in 2π units of time.

1. What is the angle between the object's position vector and the positive x-axis when $t = .25$?

   .25. (When $t = 2\pi$ it has completed one circle, through an angle of 2π, so generally $\theta = t$.)

2. At what time $t > 0$ is the angle between the object's position vector and the positive x-axis first equal to $\frac{4\pi}{3}$?

   $t = \frac{4\pi}{3}$

3. What is the angle $\theta(t)$ between the object's position vector and the positive x-axis at time t?

   $\theta(t) = t$.

4. What is the object's position vector $\vec{f}(t)$ at time t?

   $\langle \cos(t), \sin(t) \rangle$.

A *vector valued function* $\vec{r}(t)$ is a function that takes a real number t to a vector $\vec{r}(t)$. If the range consists of vectors in $\mathbb{R}^3$, for example, we write $\vec{r}:\mathbb{R}\to\mathbb{R}^3$. We may say "$\vec{r}$ maps $\mathbb{R}$ to $\mathbb{R}^3$."

$$\underbrace{\vec{r}}_{\text{function}} : \underbrace{\mathbb{R}}_{\text{contains domain}} \to \underbrace{\mathbb{R}^3}_{\text{contains range}}$$

Example: If $\vec{r}(t)=\langle\cos(t),\sin(t)\rangle$, then $\vec{r}:\mathbb{R}\to\mathbb{R}^2$.
The domain of $\vec{r}$ is $\mathbb{R}$ and the range of $\vec{r}$ is the unit circle in $\mathbb{R}^2$

You may recall that for $f:\mathbb{R}\to\mathbb{R}$ we say:

$\lim_{x\to a} f(x)=L$ means for every $\varepsilon>0$ [desired output accuracy] there is a $\delta>0$ [required input accuracy] such that, for every x,

$$\underbrace{|x-a|<\delta \;\&\; x\neq a}_{\text{within input accuracy}} \implies \underbrace{|f(x)-L|<\varepsilon}_{\text{within output accuracy}}.$$

If $\vec{r}:\mathbb{R}\to\mathbb{R}^n$, we say something very similar:

Definition: $\lim_{t\to a}\vec{r}(t)=\vec{L}$ means for every $\varepsilon>0$ [desired output accuracy] there is a $\delta>0$ [required input accuracy] such that, for every t,

$$\underbrace{\overbrace{|t-a|}^{\text{distance between } t \text{ and } a} < \delta \;\;\&\;\; t\neq a}_{\text{within input accuracy}} \implies \underbrace{\overbrace{|\vec{r}(t)-\vec{L}|}^{\text{distance between } \vec{r}(t) \text{ and } \vec{L}} < \varepsilon}_{\text{within output accuracy}}.$$

This is also like our definition of limit of a sequence, if you stretch your imagination to say "large" means "approximately infinity," and "greater than N" means "approximately infinity, to within a given accuracy."

$\lim_{n\to\infty} a_n = L$ means for every $\varepsilon>0$ [desired output accuracy] there is an N [required input accuracy] such that, for every n,

$$\underbrace{n>N}_{\text{within input accuracy}} \implies \underbrace{|a_n-L|<\varepsilon}_{\text{within output accuracy}}.$$

(Note, we don't have to say "$n>N \;\&\; n\neq\infty$," because n denotes a natural number, and ∞ is not a natural number.)

In practice, we don't often use this formal definition of limit to compute limits, although we may use it to prove things.

Theorem: If $\vec{r}(t) = \langle r_x(t), r_y(t), r_z(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} r_x(t),\, \lim_{t \to a} r_y(t),\, \lim_{t \to a} r_z(t) \right\rangle .$$

Example:

$$\lim_{t \to 0} \left\langle \frac{\sin^2 t}{t},\, \frac{\tan(\theta + t) - \tan\theta}{t} \right\rangle =$$

$$\left\langle \underbrace{\lim_{t \to 0} \frac{\sin^2 t}{t}}_{\text{l'Hopital's rule}},\, \underbrace{\lim_{t \to 0} \frac{\tan(\theta + t) - \tan(\theta)}{t}}_{\text{definition of derivative}} \right\rangle =$$

$$\left\langle \lim_{t \to 0} \frac{2 \sin t \cos t}{1},\, \frac{d}{d\theta} \tan(\theta) \right\rangle = \left\langle 0, \sec^2(\theta) \right\rangle .$$

Definition: A vector function $\vec{r}(t)$ is continuous at a if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$.

Definition: If a curve γ is the range of a vector function $\vec{r}$, we say that $\vec{r}$ parametrizes γ, or is a parametrization of γ. The t in $\vec{r}(t)$ is a parameter — different values of t give different points on γ. You can think of picking up the real number line or a part of it (the domain of $\vec{r}$), stretching, shrinking, and twisting it, and gluing it to γ, so $\vec{r}(t)$ is the place on γ where t on the number line is glued.

You can also think of $\vec{r}(t)$ as the position at time t of a point moving along γ.

Example: The function $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ parametrizes the line through $\vec{r}_0$ parallel to $\vec{v}$. To parametrize the entire line, our domain must be all of $\mathbb{R}$. If we think of $\vec{r}$ as a position function, as t goes from $-\infty$ to ∞, we traverse the entire line once.

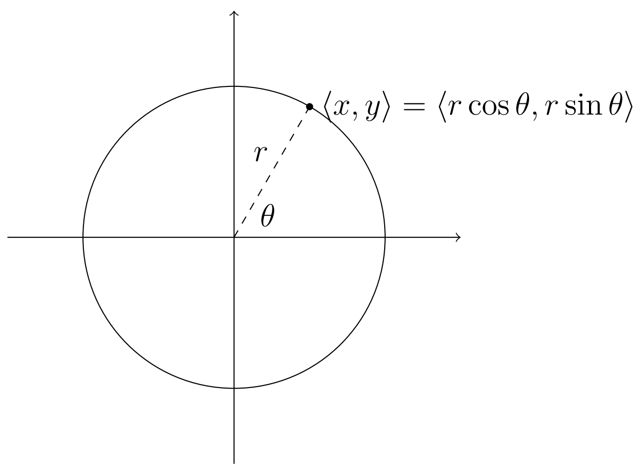
Example: The function $\vec{r}(t) = \langle a\cos(t),\, a\sin(t) \rangle$ parametrizes the circle in $\mathbb{R}^2$ with center $(0, 0)$ and radius a. To parametrize the entire circle, our domain could be $[0, 2\pi]$. If we take our domain as $\mathbb{R}$, and think of $\vec{r}$ as a position function, as t goes from $-\infty$ to ∞, we traverse the circle repeatedly.

If we take our domain to be $[0, \pi]$, we parametrize the top half of the circle.

Note: Our parametrizations give a direction to the curve. In the above examples, the line is parametrized in the direction of $\vec{v}$, and the circle is parametrized in the counterclockwise direction. A curve with an assigned direction is an *oriented* curve. The unit circle can be oriented clockwise or counterclockwise.

We prefer our parametrizations to be continuous.

Example: The function $\vec{f}(\theta) = \langle r\cos\theta, r\sin\theta\rangle$ parametrizes the circle of radius r centered at $(0,0)$ in $\mathbb{R}^2$.



Note: The numbers (r, θ) are the *polar coordinates* of the point whose usual (rectangular, or Cartesian) coordinates are (x, y). In this example, r is constant but θ changes.

The distance from the origin to the point is r, and the angle around counterclockwise from the positive x-axis to the position vector of the point is θ. We can write

$$x = r\cos\theta \quad y = r\sin\theta \quad r = \sqrt{x^2 + y^2}.$$

Example: Give parametrizations of the following curves:

1. The intersection of the paraboloid $z = x^2 + y^2$ and the plane $x = 1$.

2. The ellipse $x^2 + 4y^2 = 4$ in $\mathbb{R}^2$.

3. The intersection of the sphere $x^2+y^2+z^2=4$ with the plane $x=y+1$.

4. The intersection of the surfaces $x^4+y^4=1$ and $z=y^2-x^2$.

Example: Sketch and/or describe the curves parametrized by the following functions:

1. $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

2. $\vec{r}(t) = \langle t, t, t^2 \rangle$.

Exercise: Parametrize and sketch the curve that lies in the cone $z = \sqrt{x^2 + y^2}$ and whose projection onto the xy-plane is parametrized by $\vec{r}(t) = \langle t\cos(t),\, t\sin(t)\rangle$ for $t \geq 0$.

Hint for sketch: First sketch the projection in the xy-plane.

Exercise: Sketch or completely describe the curve parameterized by the function
$\vec{r}(t) = 2\cos(t)\left\langle \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right\rangle + 2\sin(t)\left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right\rangle$. You may notice that $\left\langle \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right\rangle$ are unit vectors and are perpendicular to each other.

Some exercises from last time:

Exercise: Find a linear equation for the plane through the origin that is parallel to both the lines $\vec{r} = \langle -1, -2, 1\rangle + t\langle 1, -1, 0\rangle$ and $\vec{r} = \langle 1, 0, 1\rangle + t\langle 1, 1, -1\rangle$.

Remark: The vectors $\langle 1, -1, 0\rangle$ and $\langle 1, 1, -1\rangle$ are parallel to the plane, so their cross product $\langle 1, 1, 2\rangle$ is normal to the plane. The origin $\langle 0, 0, 0\rangle$ is a point on the plane, so an equation for the plane is $x + y + 2z = 0$.

Exercise: Find the distance between the parallel planes

$$x + 2y - z = 4$$

$$2x + 4y - 2z = 4.$$

Remark: You can find the distance from any point on one plane to the other plane, using either of the method we used to do a similar problem in class. You can also find the equation of a line ℓ normal to both planes, say a line through the origin. The (perpendicular) distance between the planes is the distance between the points where ℓ intersects the planes; we saw in class how to find the point where a line intersects a plane. These points are $\left\langle \frac{2}{3}, \frac{4}{3}, \frac{-2}{3} \right\rangle$ and $\left\langle \frac{1}{3}, \frac{2}{3}, \frac{-1}{3} \right\rangle$, and the distance between them is $\sqrt{\frac{2}{3}}$.

Exercise: Does the line through the points $(7, 9, 3)$ and $(-2, -3, 0)$ intersect the line through the points $(2, 2, 3)$ and $(0, 0, -1)$?

Remark: You can use any of the methods given in the problem. The lines do intersect, at the point $(1, 1, 1)$.

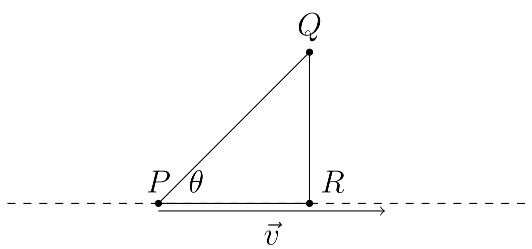
Exercise: Find the distance between the skew lines (lines that are not parallel but do not meet) $\vec{r} = \langle -1, -2, 1\rangle + t\langle 1, -1, 0\rangle$ and $\vec{r} = \langle 1, 0, 1\rangle + t\langle 1, 1, -1\rangle$.

Remark: A method other than the one mentioned in the problem: Find a vector $\vec{n}$ normal to both lines. (Take the cross product of vectors parallel to the lines.) The plane π_1 with normal vector $\vec{n}$ containing the first line (use any point on the line as a point on the plane), and the plane π_2 with normal vector $\vec{n}$ containing the second line, are two parallel planes. The distance between π_1 and π_2 is the distance between the lines, and we know how to find the distance between two planes. This distance is $\frac{4}{\sqrt{6}}$.

Remark: Do not try to memorize the various formulas for the distances between a point and a line, between a point and a plane, between two lines, between a line and a plane, between two planes. If you need to find the distance between a point and a line, for example, you should be able to figure it out. (See the next problem.)

Exercise: Think of at least two methods to find the distance between a point and a line (in three dimensions).

This picture may suggest one method. We want to find the distance between the point Q and the line through point P in the direction of vector $\vec{v}$.



Here we know the points P and Q and the vector $\vec{v}$. The pictured triangle has a right angle at R. We want the distance between Q and R. We do not know the point R.

Some Solutions: (There are probably still more.)

Method 1: The vector $\overrightarrow{PR}$ is the projection of $\overrightarrow{PQ}$ along the vector $\vec{v}$. Use this to find $\overrightarrow{PR}$. Find $\overrightarrow{RQ}$ as $\overrightarrow{PQ} - \overrightarrow{PR}$. The distance we want is the magnitude of $\overrightarrow{RQ}$.

Method 2: The distance between P and R is the absolute value of the component of $\overrightarrow{PQ}$ along $\vec{v}$. Find this, and find the distance between P and Q. Use the Pythagorean Theorem to find the distance between Q and R.

Method 3: The distance we want is $\left|\overrightarrow{PQ}\right| \sin\theta$, which equals $\dfrac{|\overrightarrow{PQ} \times \vec{v}|}{|\vec{v}|}$.

Method 4: If we say $\vec{p}$ is the position vector of P, then a point on ℓ is $\vec{p} + t\vec{v}$. Define the function $f(t)$ to be the distance between Q and $\vec{p} + t\vec{v}$; use the coordinates of P, Q, and $\vec{v}$ to find an expression for $f(t)$. Then use calculus to find the minimum value of $f(t)$. This uses the fact that R is the point on ℓ that is closest to Q.

Method 5: The point R is the point on ℓ satisfying $\overrightarrow{PR} \cdot \overrightarrow{RQ} = 0$. Use the coordinates of P, Q, and $\vec{v}$, and the expression $R = \vec{p} + t\vec{v}$, to rewrite $\overrightarrow{PR} \cdot \overrightarrow{RQ} = 0$ as a linear equation with variable t, and solve for t. Plugging in to $R = \vec{p} + t\vec{v}$ gives R. Now find the distance between R and Q.

Method 6: A different way to find R is to find an equation for the plane π that contains Q and is normal, or perpendicular, to ℓ. The point R is the point where ℓ and π intersect.

Method 7: (This is one you don't yet have the tools for. You can acquire those tools in a more advanced math class, such as linear algebra, or in some physics, engineering, or computer science classes.) Apply a transformation T that rotates around the origin to make $\vec{v}$ parallel to the x-axis. In the transformed picture, $T(R)$ has the same y- and z-coordinates as $T(P)$, and the same x-coordinate as $T(Q)$, so we know all three points.