Math 8
Winter 2020
Section 1
February 10, 2020

First, some important points from the last class:

A *vector valued function* $\vec{r}(t)$ is a function that takes a real number t to a vector $\vec{r}(t)$. If the range consists of vectors in $\mathbb{R}^3$, for example, we write $\vec{r} : \mathbb{R} \to \mathbb{R}^3$. We may say "$\vec{r}$ maps $\mathbb{R}$ to $\mathbb{R}^3$."

$$\underbrace{\vec{r}}_{\text{function}} : \underbrace{\mathbb{R}}_{\text{contains domain}} \to \underbrace{\mathbb{R}^3}_{\text{contains range}}$$

Definition: $\lim_{t\to a} \vec{r}(t) = \vec{L}$ means for every $\varepsilon > 0$ [desired output accuracy] there is a $\delta > 0$ [required input accuracy] such that, for every t,

$$\underbrace{|t-a| < \delta \ \&\ t \neq a}_{\text{within input accuracy}} \implies \underbrace{\overbrace{|\vec{r}(t) - \vec{L}|}^{\text{distance between } \vec{r}(t) \text{ and } \vec{L}} < \varepsilon}_{\text{within output accuracy}}.$$

Theorem: If
$$\vec{r}(t) = \langle r_x(t), r_y(t), r_z(t)\rangle,$$
then
$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} r_x(t),\ \lim_{t\to a} r_y(t),\ \lim_{t\to a} r_z(t)\right\rangle.$$

Definition: A vector function $\vec{r}(t)$ is continuous at a if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

Definition: The vector function $\vec{r}(t)$ parametrizes the curve γ if γ is the range of $\vec{r}(t)$.

Methods to try for parametrizing curves given as intersections of surfaces:

I. Eliminate variables by using the equations of the surfaces. Examples: Write y and z in terms of x, set $x = t$. Write z in terms of x and y, then parametrize the projection of the curve on the xy-plane.

II. If you have an equation of the form $A^2 + B^2 = 1$, where A and B are expressions involving two variables, set $A = \cos(t)$ and $B = \sin(t)$.

III. Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ for curves circling the origin in $\mathbb{R}^2$, by writing r in terms of θ, and setting $\theta = t$.

Preliminary Homework Assignment

1. If an object travels at constant velocity $\vec{v}$ between times t_1 and t_2, and we write $\vec{v} = V\vec{u}$, where V is a positive scalar and $\vec{u}$ is a unit vector:

   (a) The object's speed is $\boxed{V}$

   (b) A unit vector in the direction the object is moving is $\boxed{\vec{u}}$

   (c) The length of time the object travels is $\boxed{t_2 - t_1}$

   (d) The distance the object travels is $\boxed{V(t_2 - t_1)}$

   (e-f) The object's displacement is $\boxed{V(t_2 - t_1)\vec{u}} = \boxed{(t_2 - t_1)\vec{v}}$

2. An object travels at constant velocity between times t_1 and t_2.

   (a) If its velocity is $\vec{v}$, then its displacement is $\boxed{\vec{d} = (t_2 - t_1)\vec{v}}$

   (b) If its displacement is $\vec{d}$, then its velocity is $\boxed{\vec{v} = \frac{1}{t_2 - t_1}\vec{d}}$

3. An object travels around the unit circle in the xy-plane, and its position at time t is the point $(\cos t, \sin t)$ (with time and distance measured in your favorite units).

   (a) The object's displacement between times t and $t + \Delta t$ is
   $\langle \cos(t+\Delta t), \sin(t+\Delta t)\rangle - \langle \cos(t), \sin(t)\rangle =$
   $\boxed{\langle \cos(t+\Delta t) - \cos(t), \sin(t+\Delta t) - \sin(t)\rangle}$

   (b) If Δt is small enough, it is not a bad approximation to suppose that between times t and Δt, the object is traveling at constant velocity.
   The object's velocity between times t and Δt is approximately (using (2)(b))
   $\frac{1}{\Delta t}\langle \cos(t+\Delta t) - \cos(t), \sin(t+\Delta t) - \sin(t)\rangle =$
   $\boxed{\left\langle \frac{\cos(t+\Delta t) - \cos(t)}{\Delta t}, \frac{\sin(t+\Delta t) - \sin(t)}{\Delta t} \right\rangle}$

   (c) The object's instantaneous velocity at time t is (letting $\Delta t \to 0$)
   $\left\langle \frac{d\cos(t)}{dt}, \frac{d\sin(t)}{dt} \right\rangle = \boxed{\langle -\sin(t), \cos(t)\rangle}$

Definition: The derivative of a vector function is defined by

$$\frac{d}{dt}\vec{r}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\vec{r}(t + \Delta t) - \vec{r}(t) \right).$$

Theorem: If

$$\vec{r}(t) = \langle r_x(t), r_y(t), r_z(t) \rangle,$$

then

$$\frac{d}{dt}\vec{r}(t) = \left\langle \frac{d}{dt} r_x(t),\ \frac{d}{dt} r_y(t),\ \frac{d}{dt} r_z(t) \right\rangle.$$

Example:

$$\frac{d}{dt} \langle \cos(t),\ \sin(t) \rangle = \langle -\sin(t),\ \cos(t) \rangle.$$

If $\vec{r}(t)$ denotes the position of a moving object at time t, then the derivative $\vec{r}\,'(t)$ denotes the velocity of that object, its magnitude $|\vec{r}\,'(t)|$ is the object's speed, and the unit vector $\vec{T}(t)$ in the direction of $\vec{r}\,'(t)$ gives the direction of motion.

The unit vector $\vec{T}(t)$ is tangent to the object's path, and is called the unit tangent vector. The direction of the unit tangent vector is the direction in which the curve is oriented.

Definition:

$$\int \langle r_x(t), r_y(t), r_z(t) \rangle \; dt = \left\langle \int r_x(t)\, dt,\ \int r_y(t)\, dt,\ \int r_z(t)\, dt \right\rangle$$

$$\int_a^b \langle r_x(t), r_y(t), r_z(t) \rangle \; dt = \left\langle \int_a^b r_x(t)\, dt,\ \int_a^b r_y(t)\, dt,\ \int_a^b r_z(t)\, dt \right\rangle$$

Theorem:

$$\int_a^b \vec{r}\,'(t)\, dt = \vec{r}(b) - \vec{r}(a).$$

If $\vec{r}(t)$ is the position of a moving object at time t then this is the net displacement between times $t = a$ and $t = b$.

Integrate velocity to find displacement.

Example: An object moves in the plane with position function

$$\vec{r}(t) = \langle 1 + \cos(t), \sin(t) \rangle .$$

Find a vector parametric equation for the line tangent to the object's path at the point $\left\langle \frac{3}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

$\left\langle \frac{3}{2}, \frac{\sqrt{3}}{2} \right\rangle = \vec{r}\left(\frac{\pi}{3}\right)$ is the object's position at time $t = \frac{\pi}{3}$. This is a point on the tangent line.

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$ is the object's velocity at time t.

$\vec{r}\,'\left(\frac{\pi}{3}\right) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ is the object's velocity at time $t = \frac{\pi}{3}$. Since velocity points in the direction of motion, this is a vector in the direction of the tangent line.

$\langle x, y \rangle = \left\langle \frac{3}{2}, \frac{\sqrt{3}}{2} \right\rangle + t \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ is an equation for the tangent line.

The function $\vec{\ell}(t) = \left\langle \frac{3}{2}, \frac{\sqrt{3}}{2} \right\rangle + t \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ parametrizes the tangent line.

Notice, the function $\vec{\ell}(t)$ represents the motion of an object traveling along the tangent line at the same velocity our original object has at time $t = \frac{\pi}{3}$, starting at time $t = 0$ at the position of our original object at time $t = \frac{\pi}{3}$.

If we want the object with position function ℓ to be at the same point as our original object at time $t = \frac{\pi}{3}$, we should adjust its clock. We can do this by replacing t with $t - \frac{\pi}{3}$, to get a new position function
$\vec{f}(t) = \left\langle \frac{3}{2}, \frac{\sqrt{3}}{2} \right\rangle + \left(t - \frac{\pi}{3}\right) \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$.

You can check this function has the same value and derivative as $\vec{r}(t)$ when $t = \frac{\pi}{3}$.

It is the *tangent approximation* to $\vec{r}(t)$ near $t_0 = \frac{\pi}{3}$. It can be written as

$$\vec{f}(t) = \vec{r}\,(t_0) + (t - t_0)\, \vec{r}\,'\,(t_0)\,,$$

which should look familiar.

Theorem: If all functions mentioned are differentiable, then

$$\frac{d}{dt}(a\vec{r}(t) + b\vec{p}(t)) = a\vec{r}\,'(t) + b\vec{p}\,'(t)$$

This combines the constant multiple and sum rules into a linear combination rule.

$$\frac{d}{dt}\left(f(t)\vec{r}(t)\right) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{p}(t)\right) = \vec{r}\,'(t) \cdot \vec{p}(t) + \vec{r}(t) \cdot \vec{p}\,'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t) \times \vec{p}(t)\right) = \vec{r}\,'(t) \times \vec{p}(t) + \vec{r}(t) \times \vec{p}\,'(t)$$

These are different versions of the product rule (order matters in the cross product one). The final fact is a version of the chain rule:

$$\frac{d}{dt}(\vec{r}(f(t)) = f'(t)\vec{r}\,'(f(t))$$

If we write $u = f(t)$ and $\vec{v} = \vec{r}(f(t)) = \vec{r}(u)$, we can rewrite this as

$$\frac{d\vec{v}}{dt} = \frac{du}{dt}\frac{d\vec{v}}{du}.$$

Theorem: If $\vec{r}(t)$ is differentiable, then $|\vec{r}(t)|$ is constant if and only if $\vec{r}(t) \perp \vec{r}\,'(t)$ for all t.

Proof: We know $|\vec{r}(t)|$ is constant if and only if $|\vec{r}(t)|^2$ is constant, and $|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$. A function is constant if and only if its derivative is always zero, so $|\vec{r}(t)|^2$ is constant if and only if

$$0 = \frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 2(\vec{r}(t) \cdot \vec{r}\,'(t)).$$

We also know that $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$ if and only if $\vec{r}(t) \perp \vec{r}\,'(t)$.

Putting all this together, $|\vec{r}(t)|$ is constant if and only if $\vec{r}(t) \perp \vec{r}\,'(t)$ for all t.

Definition: If a curve γ is parametrized by a vector function $\vec{r}$, and $\vec{r}\,'(t)$ is defined and nonzero for every t (except possibly end points of the domain), then $\vec{r}$ is a regular parametrization, or smooth parametrization, of γ. A curve with a regular parametrization is called a smooth curve.

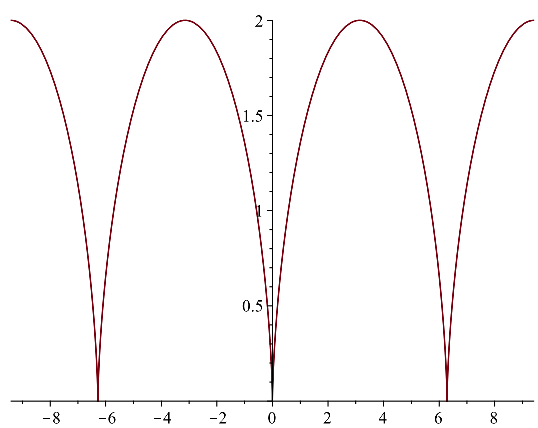
Example: The curve parametrized by

$\vec{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$ for $-3\pi \leq t \leq 3\pi$ is not smooth, because although $\vec{r}\,'(t) = \langle 1 - \cos t, \sin t\rangle$ is defined everywhere in the domain, it equals $\vec{0}$ at some points ($t = 2n\pi$).

This is a cycloid, the path traveled by a point on the edge of a wheel as it rolls along on the x-axis. (This picture is distorted; the vertical axis is stretched out.) There are some sharp corners on this path, where the point has zero velocity for an instant as it abruptly changes direction.

At those points we do not have a unit tangent vector.

This curve is *piecewise smooth* because it can be broken up into finitely many smooth pieces.



Exercise: The position function of a moving object is

$$\vec{r}(t) = \langle 1, 3\cos(t), 3\sin(t) \rangle .$$

1. Describe the object's path as completely as possible.

2. Find the object's velocity in general and at time $t = \dfrac{\pi}{2}$.

3. Find an equation for the line tangent to the object's path at the point $\langle 1, 0, 3 \rangle$.

4. Find $\displaystyle\int_0^{\pi} \vec{r}\,'(t)\, dt$. (What does this represent?)

5. Find the distance between the object's position at $t = 0$ and the object's position at $t = \pi$. (How is this related to your answer to part (4)?)

6. Using part (1), find the length of the curve the object travels along between time $t = 0$ and time $t = \pi$. (Don't try to use calculus for this one. Use geometry.)

Exercise: If the position of a moving object at time t is

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle ,$$

what can we say about the motion of that object?

Once you have said everything else you have to say: What is the projection of the object's path on the xy-plane? What is the angle between the object's velocity vector at time t and the xy-plane? (Think about this one. It requires some cleverness. Is the answer different for different values of t?) Given that, can you make a guess about the length of the path the object travels along between $t = 0$ and $t = 2\pi$?

We will do some deeper analysis later.

Some examples from last time:

Example: Parametrize the intersection of the sphere $x^2 + y^2 + z^2 = 4$ with the plane $x = y + 1$.

We can begin by eliminating one variable. Substitute $x = y + 1$ into the other equation to get

$$(y+1)^2 + y^2 + z^2 = 4.$$

This is the projection of our curve onto the yz-plane. By parametrizing this, we can express y and z in terms of t. We begin by doing some algebra, including completing the square:

$$y^2 + 2y + 1 + y^2 + z^2 = 4$$

$$2(y^2 + y) + z^2 = 3$$

$$2((y + \frac{1}{2})^2 - \frac{1}{4}) + z^2 = 3$$

$$2(y + \frac{1}{2})^2 + z^2 = \frac{7}{2}$$

$$\frac{4}{7}(y + \frac{1}{2})^2) + \frac{2}{7}z^2 = 1$$

$$\left(\frac{2}{\sqrt{7}}y + \frac{1}{\sqrt{7}}\right)^2 + \left(\frac{\sqrt{2}}{\sqrt{7}}z\right)^2 = 1.$$

Now we can use $\cos^2 t + \sin^2 t = 1$, and set

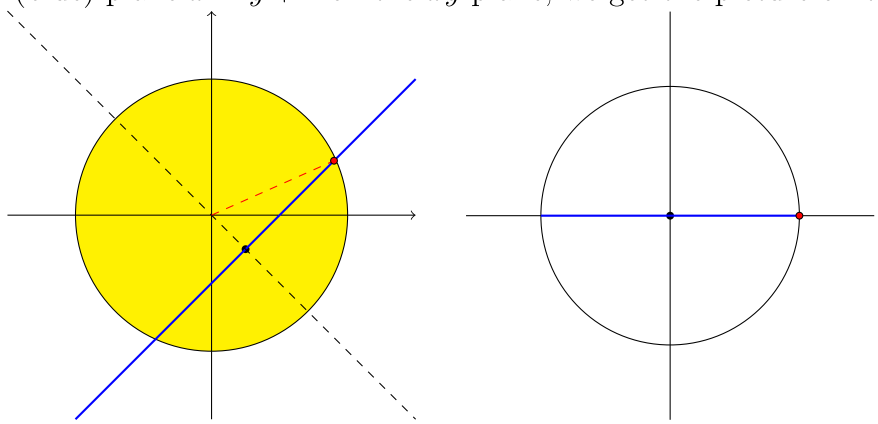
$$\left(\frac{2}{\sqrt{7}}y + \frac{1}{\sqrt{7}}\right) = \cos t \qquad \left(\frac{\sqrt{2}}{\sqrt{7}}z\right) = \sin t$$

$$y = \frac{\sqrt{7}}{2}\cos t - \frac{1}{2} \qquad z = \frac{\sqrt{7}}{\sqrt{2}}\sin t \qquad x = y + 1 = \frac{\sqrt{7}}{2}\cos t - \frac{1}{2}$$

$$\boxed{\vec{f}(t) = \left\langle \frac{\sqrt{7}}{2}\cos t + \frac{1}{2},\ \frac{\sqrt{7}}{2}\cos t - \frac{1}{2},\ \frac{\sqrt{7}}{\sqrt{2}}\sin t \right\rangle.}$$

We can also look at this example geometrically. The intersection of a sphere and a plane is a circle, in this case a circle in the plane $x = y + 1$.

Looking at the projections of the (yellow) sphere $x^2 + y^2 + z^2 = 4$ and the (blue) plane $x = y + 1$ on the xy-plane, we get the picture on the left:



The portion of the blue line that lies inside the yellow disc is a diameter of the circle we are looking for, and the black dot is the center of that circle.

By the symmetry of the picture, the center of the circle is on the dashed black line $x = -y$. Therefore, it is the intersection of the two lines $x = -y$ and $x = y + 1$, or $(\frac{1}{2}, -\frac{1}{2})$. (In three dimensions, $(\frac{1}{2}, -\frac{1}{2}, 0)$.)

The distance from the origin to the black dot is $\frac{1}{\sqrt{2}}$. The distance from the origin to the red dot is the radius of the sphere, or 2. The Pythagorean theorem tells us that the distance between the black dot and the red dot is $\frac{\sqrt{7}}{\sqrt{2}}$; this is the radius of our circle. The vector between the black dot and the red dot is parallel to the line $x = y$; a unit vector in that direction is $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, or in three dimensions $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$.

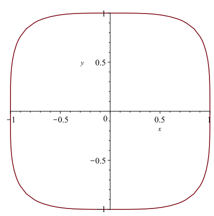
In the picture on the right, we see the intersection of the sphere with the plane $x = y + 1$, which is the circle we want to parametrize. The black dot is the point $(\frac{1}{2}, -\frac{1}{2}, 0)$. A unit vector in the horizontal direction is the unit vector pointing from the black dot to the red dot, $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$. A unit vector in the vertical direction is $\langle 0, 0, 1 \rangle$. The radius of the circle is $\frac{\sqrt{7}}{\sqrt{2}}$.

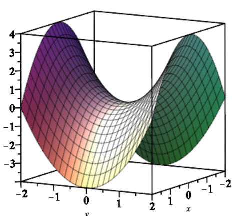
In general, we can parametrize a circle with center $\vec{r}_0$ and radius r, in a plane parallel to unit vectors $\vec{v}$ and $\vec{w}$, by $\vec{f}(t) = \vec{r}_0 + r(\cos t)\vec{v} + r(\sin t)\vec{w}$. In our case, this becomes

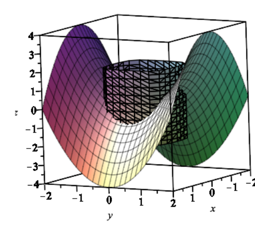
$$\vec{f}(t) = \left\langle \frac{1}{2}, -\frac{1}{2}, 0 \right\rangle + \frac{\sqrt{7}}{\sqrt{2}}(\cos t) \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle + \frac{\sqrt{7}}{\sqrt{2}}(\sin t) \langle 0, 0, 1 \rangle .$$

Example: Parametrize the intersection of the surfaces $x^4 + y^4 = 1$ and $z = y^2 - x^2$.

The curve in the xy-plane $x^4 + y^4 = 1$ is pictured at the left. The surface $z = y^2 - x^2$ is the saddle shown in the center. In the picture at right, this surface is intersected with the surface $x^4 + y^4 = 1$ in three dimensions.







Since z does not appear in $x^4 + y^4 = 1$, this is the projection of our curve onto the xy-plane. We can parametrize this curve, then use the equation $z = y^2 = x^2$ to add the z-coordinate.

To parametrize this curve, which circles the origin in the xy-plane, we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. The goal is to write r in terms of θ, and then set $\theta = t$ to get the parametrization:

$$x^4 + y^4 = 1 \qquad r^4\cos^4\theta + r^4\sin^4\theta = 1$$

$$r^4 = (\cos^4\theta + \sin^4\theta =)^{-1} \qquad r = (\cos^4\theta + \sin^4\theta)^{-\frac{1}{4}}$$

(For the last step, we use $r \geq 0$.) Now, setting $\theta = t$, our parametrization is

$$x = r\cos\theta = (\cos^4 t + \sin^4 t)^{-\frac{1}{4}}\cos t \qquad y = r\sin\theta = (\cos^4 t + \sin^4 t)^{-\frac{1}{4}}\sin t$$

$$z = y^2 - x^2 = (\cos^4 t + \sin^4 t)^{-\frac{1}{2}}(\sin^2 t - \cos^2 t). \qquad \vec{f}(t) =$$

$$\left\langle (\cos^4 t + \sin^4 t)^{-\frac{1}{4}}\cos t,\ (\cos^4 t + \sin^4 t)^{-\frac{1}{4}}\sin t,\ (\cos^4 t + \sin^4 t)^{-\frac{1}{2}}(\sin^2 t - \cos^2 t) \right\rangle.$$