Math 8
Winter 2020
Section 1
February 12, 2020

First, some important points from the last class:

Definition: The derivative of a vector function is defined by

$$\frac{d}{dt}\vec{r}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\vec{r}(t+\Delta t) - \vec{r}(t)\right).$$

Theorem: If

$$\vec{r}(t) = \langle r_x(t), r_y(t), r_z(t) \rangle,$$

then

$$\frac{d}{dt}\vec{r}(t) = \left\langle \frac{d}{dt}r_x(t),\ \frac{d}{dt}r_y(t),\ \frac{d}{dt}r_z(t) \right\rangle.$$

If $\vec{r}(t)$ denotes the position of a moving object at time t, then the derivative $\vec{r}\,'(t)$ denotes the velocity of that object, its magnitude $|\vec{r}\,'(t)|$ is the object's speed, and the unit vector $\vec{T}(t)$ in the direction of $\vec{r}\,'(t)$ gives the direction of motion.

The unit vector $\vec{T}(t)$ is tangent to the object's path, and is called the unit tangent vector.

Theorem: If all functions mentioned are differentiable, then

$$\frac{d}{dt}(a\vec{r}(t) + b\vec{p}(t)) = a\vec{r}\,'(t) + b\vec{p}\,'(t)$$

$$\frac{d}{dt}\left(f(t)\vec{r}(t)\right) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{p}(t)\right) = \vec{r}\,'(t)\cdot\vec{p}(t) + \vec{r}(t)\cdot\vec{p}\,'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t)\times\vec{p}(t)\right) = \vec{r}\,'(t)\times\vec{p}(t) + \vec{r}(t)\times\vec{p}\,'(t)$$

$$\frac{d}{dt}(\vec{r}(f(t)) = f'(t)\vec{r}\,'(f(t)) \quad \text{or} \quad \frac{d\vec{r}}{dt} = \frac{du}{dt}\frac{d\vec{r}}{du}$$

Theorem: If $\vec{r}(t)$ is differentiable, then $|\vec{r}(t)|$ is constant if and only if $\vec{r}(t) \perp \vec{r}\,'(t)$ for all t.

Definition: If a curve γ is the range of a vector function $\vec{r}$, we say that $\vec{r}$ parametrizes γ, or is a parametrization of γ. (Think of t in $\vec{r}(t)$ as a parameter — different values of t give different points on γ.)

If $\vec{r}\,'(t)$ is defined and nonzero for every t, then $\vec{r}$ is a regular parametrization, or smooth parametrization, of γ. A curve with a regular parametrization is called a smooth curve.

Definition:

$$\int \langle r_x(t), r_y(t), r_z(t)\rangle \; dt = \left\langle \int r_x(t)\,dt,\; \int r_y(t)\,dt,\; \int r_z(t)\,dt \right\rangle$$

$$\int_a^b \langle r_x(t), r_y(t), r_z(t)\rangle \; dt = \left\langle \int_a^b r_x(t)\,dt,\; \int_a^b r_y(t)\,dt,\; \int_a^b r_z(t)\,dt \right\rangle$$

Theorem:

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a).$$

If $\vec{r}(t)$ is the position of a moving object at time t then this is the net displacement between times $t = a$ and $t = b$.

If $\vec{r}(t)$ is a vector-valued function, then for t near t_0, we can approximate $\vec{r}(t)$ by

$$\vec{L}(t) = \vec{r}(t_0) + (t - t_0)\,\vec{r}\,'(t_0).$$

This is exactly like the tangent line approximation, or degree 1 Taylor polynomial, for functions from $\mathbb{R}$ to $\mathbb{R}$. It *is* the degree 1 Taylor polynomial for $\vec{r}$ centered at t_0.

If $\vec{r}$ parametrizes the curve γ, then $\vec{L}$ parametrizes the tangent line to γ at the point $\vec{r}(t_0)$. (Unless $\vec{r}\,'(t_0) = \vec{0}$, in which case $\vec{L}$ is a constant function with value $\vec{r}(t_0)$.)

If $\vec{r}$ is a position function of a moving object, then $\vec{L}$ is the position function of another object, moving with constant velocity, that has the same position and velocity as our original object at time t_0.

Example: Find a vector parametric equation for the tangent line to the curve γ parametrized by $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$ at the point $\left(\sqrt{2},\ \sqrt{2},\ \frac{\pi}{4}\right)$.

This point is $\vec{r}\left(\frac{\pi}{4}\right)$, so we should set $t_0 = \frac{\pi}{4}$, and use the tangent approximation. Since $\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), 1\rangle$, we get

$$\vec{L}(t) = \vec{r}(t_0) + (t - t_0)\,\vec{r}\,'(t_0) = \langle 2\cos(t_0), 2\sin(t_0), t_0\rangle + (t - t_0)\,\langle -2\sin(t_0), 2\cos(t_0), 1\rangle =$$

$$\left\langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4} \right\rangle + \left(t - \frac{\pi}{4}\right) \left\langle -\sqrt{2}, \sqrt{2}, 1 \right\rangle.$$

A vector parametric equation for this line is

$$\langle x, y, z\rangle = \left\langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4} \right\rangle + \left(t - \frac{\pi}{4}\right) \left\langle -\sqrt{2}, \sqrt{2}, 1 \right\rangle.$$

1. Suppose an object is traveling with variable velocity $\vec{v}(t)$ for a period of time of length Δt, and t_i is some particular time in that period. If Δt is small enough, then over that period of time

   (a) The object's displacement is approximately $\boxed{\vec{v}(t_i)\,\Delta t}$;

   (b) The distance the object travels is approximately $\boxed{|\vec{v}(t_i)|\,\Delta t}$.

2. If the object travels between times $t = a$ and $t = b$, we break that time period up into n-many small periods of time of length Δt, and for $i = 1, \ldots, n$ we choose time t_i in the i^{th} time period, then

   (a) The distance the object travels during the i^{th} time period is approximately $\boxed{|\vec{v}(t_i)|\,\Delta t}$;

   (b) The total distance the object travels is approximately $\boxed{\sum_{i=1}^{n} |\vec{v}(t_i)|\,\Delta t}$.

3. Taking a limit as $n \to \infty$, the total distance the object travels is

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} |\vec{v}(t_i)|\,\Delta t = \int_a^b |\vec{v}(t)|\,dt}.$$

   (Note, this is not the same as net distance. If an object travels once around the unit circle, the net distance it moves is 0, since it ends up where it started. However, the total distance it travels is the length of its path, which is 2π.)

We saw that if $\vec{r}(t)$ is the position of a moving object at time t, then $\vec{v}(t) = \vec{r}\,'(t)$ is its velocity, and

$$\int_a^b \vec{v}(t)\,dt = \int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a)$$

is its net displacement between times $t = a$ and $t = b$.

Example: An object travels around the unit circle with position function $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$. Its velocity is $\vec{v}(t) = \langle -\sin(t), \cos(t) \rangle$ and

$$\int_0^{2\pi} \vec{v}(t)\,dt = \int_0^{2\pi} \langle -\sin(t), \cos(t) \rangle\;dt = \left\langle \int_0^{2\pi} -\sin(t)\,dt,\; \int_0^{2\pi} \cos(t)\,dt \right\rangle = \langle 0, 0 \rangle\,.$$

The net displacement is $\vec{0}$ since the object ends up where it started, even though it travels a distance of 2π units around the unit circle.

We could find the distance it traveled along its path by integrating its *speed* (the magnitude of its velocity) with respect to time:

$$\int_0^{2\pi} |\vec{v}(t)|\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$$

Definition: If a curve γ has a regular (smooth) parametrization

$$\vec{r} : [a, b] \to \mathbb{R}^n$$

that does not retrace any portion of γ, then the arc length of γ is

$$L = \int_a^b |\vec{r}\,'(t)|\,dt.$$

The arc length function is the function

$$s : [a, b] \to [0, L]$$

that takes t to the arc length of the portion of γ between $\vec{r}(a)$ and $\vec{r}(t)$:

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du.$$

The parametrization of γ by arc length is the function

$$\vec{r} \circ s^{-1} : [0, L] \to \mathbb{R}^n$$

that takes a number s to the point on γ that is a distance of s units along γ from the starting point $\vec{r}(a)$.

We sometimes denote speed by $\dfrac{ds}{dt}$ since in fact $\dfrac{ds}{dt} = |\vec{r}\,'|$.

Example: Consider the curve γ in $\mathbb{R}^2$ defined by

$$\vec{r}(t) = \left\langle 3\cos(t^2), 3\sin(t^2) \right\rangle$$

for $0 \leq t \leq 2$. Find the length of γ, the arc length function, and the parametrization of γ by arc length.

Definition: The curvature of a curve γ with regular parametrization $\vec{r}$ at a point $\vec{r}(t)$ is the magnitude of the rate of change of the unit tangent vector $\vec{T}$ with respect to arc length,

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|.$$

We can use the chain rule:

$$\frac{d}{dt}(\vec{r}(f(t)) = f'(t)\vec{r}'(f(t))$$

or, letting $u = f(t)$,

$$\frac{d\vec{r}}{dt} = \frac{du}{dt}\frac{d\vec{r}}{du}$$

In our case:

$$\frac{d\vec{T}}{dt} = \frac{ds}{dt}\frac{d\vec{T}}{ds}.$$

$$\frac{d\vec{T}}{ds} = \frac{1}{\frac{ds}{dt}}\frac{d\vec{T}}{dt}.$$

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{\frac{ds}{dt}}\left|\frac{d\vec{T}}{dt}\right|.$$

Theorem: The curvature κ of a curve γ with regular parametrization $\vec{r}$, at a point $\vec{r}(t)$, can be calculated using any one of the following (where $'$ always means the derivative with respect to t):

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{1}{\frac{ds}{dt}}\frac{d\vec{T}}{dt}\right| = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}.$$

To get the last formula, we use the fact that since $|\vec{T}|$ is constant, we have $\vec{T} \perp \vec{T}\,'$. We have $\vec{r}\,' = |\vec{r}\,'|\vec{T} = \frac{ds}{dt}\vec{T}$. Using the product rule, $\vec{r}\,'' = \left(\frac{d^2s}{dt^2}\right)\vec{T} + \frac{ds}{dt}\vec{T}\,'$.

This gives $\vec{r}\,' \times \vec{r}\,'' = \left(\frac{ds}{dt}\vec{T}\right) \times \left(\left(\frac{d^2s}{dt^2}\right)\vec{T} + \frac{ds}{dt}\vec{T}\,'\right)$. Using the fact that $\vec{T} \times \vec{T} = \vec{0}$, we get $\vec{r}\,' \times \vec{r}\,'' = \left(\frac{ds}{dt}\right)^2(\vec{T} \times \vec{T}\,') = |\vec{r}\,'|^2(\vec{T} \times \vec{T}\,')$.

Now we use the fact that $\vec{T} \perp \vec{T}\,'$, so $|\vec{T} \times \vec{T}\,'| = |\vec{T}|\,|\vec{T}\,'| = |\vec{T}\,'|$.

This gives $|\vec{r}\,' \times \vec{r}\,''| = |\vec{r}\,'|^2(|\vec{T}\,'|)$, or $|\vec{T}\,'| = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^2}$. Now, finally,

$$\kappa = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}.$$

Example: Find the curvature of the circle of radius R around $(0,0,0)$ in the plane $z=0$.

Theorem: The arc length and curvature of a curve can be computed using any regular parametrization, and the answer will be the same.

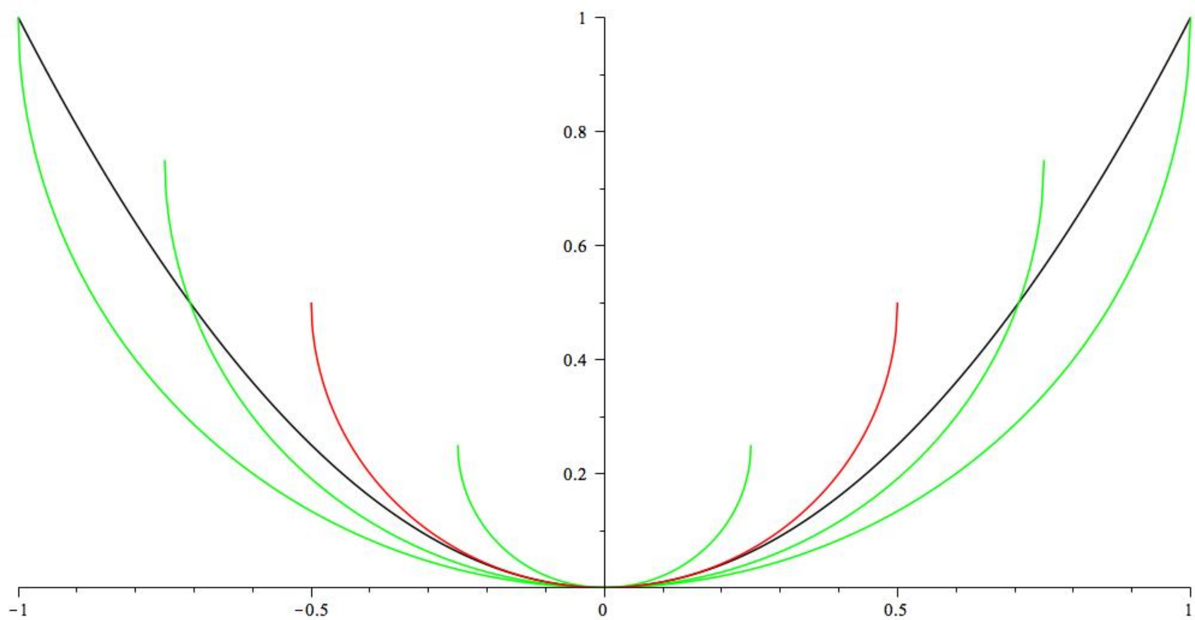
We say arc length and curvature do not depend on the parametrization.

A curve of curvature κ bends as much as a circle of radius $\frac{1}{\kappa}$.

Note: By a result we proved last time, since $\vec{T}$ is constant, we know $\vec{T} \perp \vec{T}\,'$.

Definition: The (principal) unit normal vector to a parametrized curve is $\vec{N} = \frac{1}{|\vec{T}\,'|}\vec{T}\,'$, the unit vector in the direction of $\vec{T}\,'$. It points toward the inside of the bend the curve is making.

The circle that best matches the curve γ at a given point has radius $\frac{1}{\kappa}$, and center along the ray from that point in the direction of $\vec{N}$.



The black curve is parametrized so the orientation goes from left to right.

At the origin, the unit tangent vector $\vec{T}$ points in the x-direction and the unit normal vector $\vec{N}$ points in the y-direction. The curvature of the curve at that point is $\kappa = 2$.

The circle that best matches the curve at that point has radius $\dfrac{1}{\kappa} = \dfrac{1}{2}$, and center along the ray from the origin in the direction of $\vec{N}$, or along the positive y-axis. This is the circle drawn in red.

The circles drawn in green are also tangent to the curve at the origin, but they have different curvature, so they are not as close a match.

Exercise: Find the arc length, and the curvature at any point, of the curve parametrized by

$\vec{r}(t) = \langle t, \sin(t), \cos(t) \rangle$ for $0 \leq t \leq 2\pi$. Give a parametrization of this curve by arc length.

Exercise: The curve γ is the intersection of the surface $z = x^2 - y^2$ with the plane $x = 3$. Find a vector parametric equation for the tangent line to γ at the point $(3, 1, 8)$.

Find a definite integral giving the arc length of the portion of γ where $-1 \leq y \leq 1$. You do not have to evaluate this integral.

Exercise: Suppose

$$f : [a, b] \to [0, 2\pi]$$

is any differentiable function with positive derivative at every point, $f(a) = 0$, and $f(b) = 2\pi$. Parametrize the unit circle by

$$\vec{r}(t) = \langle \cos(f(t)), \sin(f(t)) \rangle \qquad a \le t \le b.$$

Use this parametrization to compute the arc length of the unit circle.

Of course, your answer should be 2π. The point is to demonstrate why it does not matter which parametrization you choose.

Exercise: The curve parametrized by

$\vec{r}(t) = \langle 2\cos(t^2), 2\sin(t^2) \rangle$ is a circle of radius 2, so it should have curvature $\kappa = \frac{1}{2}$. Check this by computing the curvature using this parametrization. (Note that this is the same function as in the example on page 5, so you can find some of the relevant computations there.)

Some solutions from last time:

Exercise: The position function of a moving object is

$$\vec{r}(t) = \langle 1, 3\cos(t), 3\sin(t) \rangle .$$

1. Describe the object's path as completely as possible.

   This is a circle in the plane $x = 1$, with center $(1, 0, 0)$ and radius 3.

2. Find the object's velocity in general and at time $t = \dfrac{\pi}{2}$.

$$\vec{r}\,'(t) = \langle 0, -3\sin(t), 3\cos(t) \rangle \qquad \vec{r}\,'\left(\frac{\pi}{2}\right) = \langle 0, -3, 0 \rangle .$$

3. Find an equation for the line tangent to the object's path at the point $\langle 1, 0, 3 \rangle$.

   Since $\langle 1, 0, 3 \rangle = \vec{r}\left(\dfrac{\pi}{2}\right)$, we use the tangent approximation at $t_0 = \dfrac{\pi}{2}$:

$$\langle x, y, z \rangle = \vec{r}(t_0) + (t - t_0)\vec{r}\,'(t_0) =$$

$$\vec{r}\left(\frac{\pi}{2}\right) + \left(t - \frac{\pi}{2}\right)\vec{r}\,'\left(\frac{\pi}{2}\right) = \langle 1, 0, 3 \rangle + \left(t - \frac{\pi}{2}\right)\langle 0, -3, 0 \rangle .$$

4. Find $\displaystyle\int_0^{\pi} \vec{r}\,'(t)\,dt$. (What does this represent?)

$$\int_0^{\pi} \vec{r}\,'(t)\,dt = \int_0^{\pi} \langle 0, -3\sin(t), 3\cos(t) \rangle\ dt = \langle 0, -6, 0 \rangle.$$

   This is the net displacement over the time period $0 \le t \le \pi$, or $(\vec{r}(\pi) - \vec{r}(0))$.

5. Find the distance between the object's position at $t = 0$ and the object's position at $t = \pi$. (How is this related to your answer to part (4)?)

   The distance is the magnitude of the displacement, or 6. We can also directly compute the distance between $\vec{r}(0) = \langle 0, 3, 0 \rangle$ and $\vec{r}(\pi) = \langle 0, -3, 0 \rangle$

6. Using part (1), find the length of the curve the object travels along between time $t = 0$ and time $t = \pi$. (Don't try to use calculus for this one. Use geometry.)

   During the time period $0 \le t \le \pi$, the object travels halfway around a circle of radius 3, so the length of the path is half the circumference of the circle, or 3π.

   Note that this is greater than the magnitude of the net displacement.