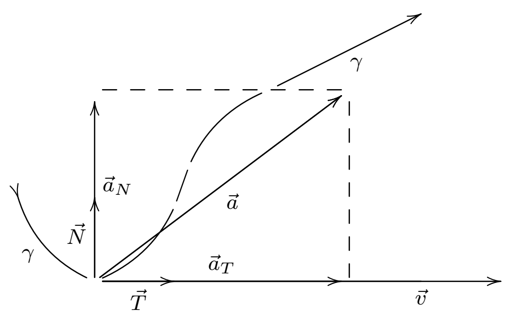
Math 8
Winter 2020
Section 1
February 17, 2020

First, some important points from the last class:

$$\vec{a} = \underbrace{\left(\frac{d^2s}{dt^2}\right)}_{a_{\mathbf{T}} \text{ tangential component}} \vec{T} + \underbrace{\left(\left(\frac{ds}{dt}\right)^2 \kappa\right)}_{\vec{a}_{\mathbf{N}} \text{ normal component}} \vec{N}$$



$$\vec{a}_{\mathbf{T}} = a_{\mathbf{T}}\vec{T} \qquad \vec{a}_{\mathbf{N}} = a_{\mathbf{N}}\vec{N} \qquad \vec{a} = \vec{a}_{\mathbf{T}} + \vec{a}_{\mathbf{N}}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \text{tangential component} = \text{linear acceleration}$$

$$a_{\mathbf{T}}\vec{T} = \text{proj}_{\vec{T}}(\vec{a}) = \text{proj}_{\vec{v}}(\vec{a})$$

$$a_{\mathbf{T}} = \text{comp}_{\vec{T}}(\vec{a}) = \text{comp}_{\vec{v}}(\vec{a})$$

$$a_{\mathbf{N}} = \left(\frac{ds}{dt}\right)^2 \kappa = \text{normal component}$$

$$a_{\mathbf{N}}\vec{N} = \text{proj}_{\vec{N}}(\vec{a}) = \vec{a} - a_{\mathbf{T}}\vec{T}$$

$$a_{\mathbf{N}} = \text{comp}_{\vec{N}}(\vec{a})$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{\left(\frac{ds}{dt}\right)^3}$$

Functions of several variables:

$$f : \mathbb{R}^n \to \mathbb{R}$$

Examples:

$$f(x,y) = x^2 + 4y^2 \qquad g(x,y,z) = x^2 + y^2 - z$$

Today we look at

1. graphs
2. level sets (level curves and level surfaces)

For $f : \mathbb{R}^2 \to \mathbb{R}$, the graph of f is a surface in $\mathbb{R}^3$, the set of all points $(x, y, f(x,y))$.

We can get an idea of the shape of the graph of f by drawing some *level curves* in $\mathbb{R}^2$, curves $f(x,y) = k$ for various values of k. If we do this for equally spaced values of k, we get something like a topographical map of the graph of f. The level curves are close together where the surface is steep.

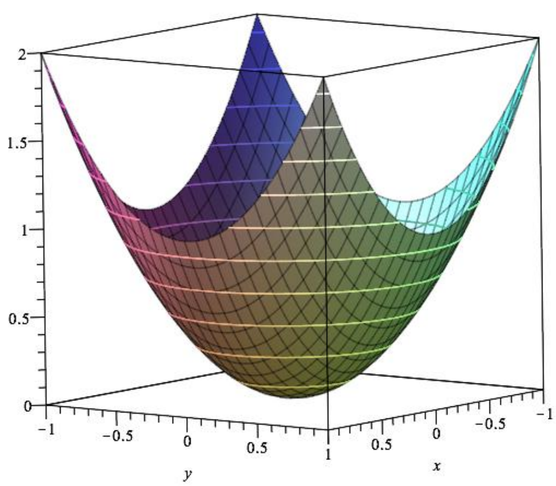
This is called a contour map, and the level curves are also called contour lines.

Example: $f(x, y) = x^2 + y^2$. The graph of f is a paraboloid $z = x^2 + y^2$ in $\mathbb{R}^3$.

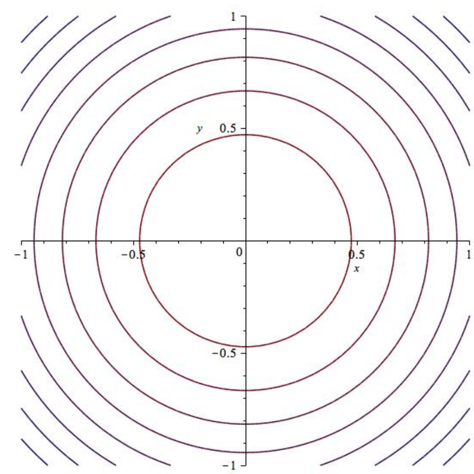
In the picture at the left, we see the graph of f, and a number of horizontal slices for equally spaced values of z. These horizontal slices are the intersections of the graph and the planes $z = k$. They are also called the *traces* of the graph on the planes $z = k$.

The level curves of f are circles $x^2 + y^2 = k$ of radius $\sqrt{k}$ in $\mathbb{R}^2$. They are drawn in the picture on the right. Another way to think of this is as the projections of the traces in the picture on the left onto the xy-plane.

Notice that while the vertical spacing between the traces is uniform, the level curves are closer together when the graph is steeper.



Graph of f in $\mathbb{R}^3$

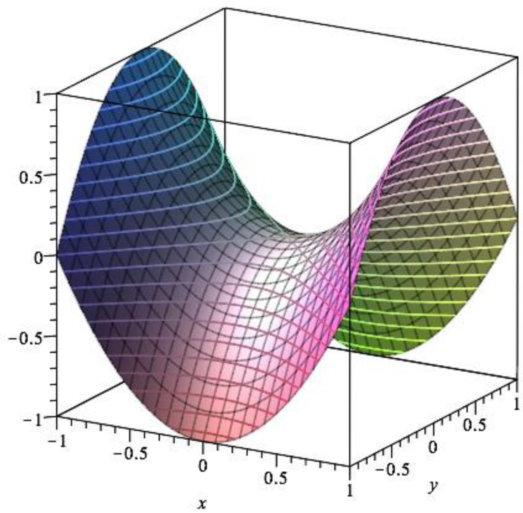


Level curves of f in $\mathbb{R}^2$

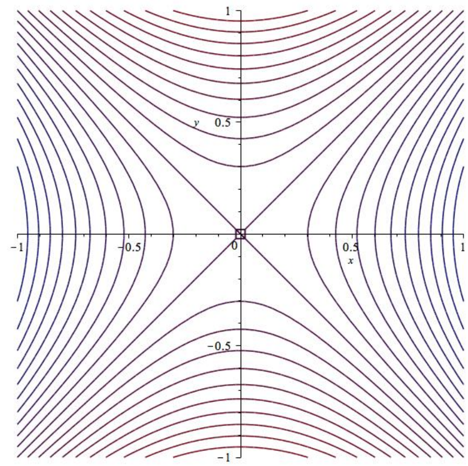
Example: $f(x, y) = x^2 - y^2$. The graph of f is a saddle $z = x^2 - y^2$ in $\mathbb{R}^3$.

The level curves of f are hyperbolae $x^2 - y^2 = k$ in $\mathbb{R}^2$.

(These pictures are drawn using Maple.)

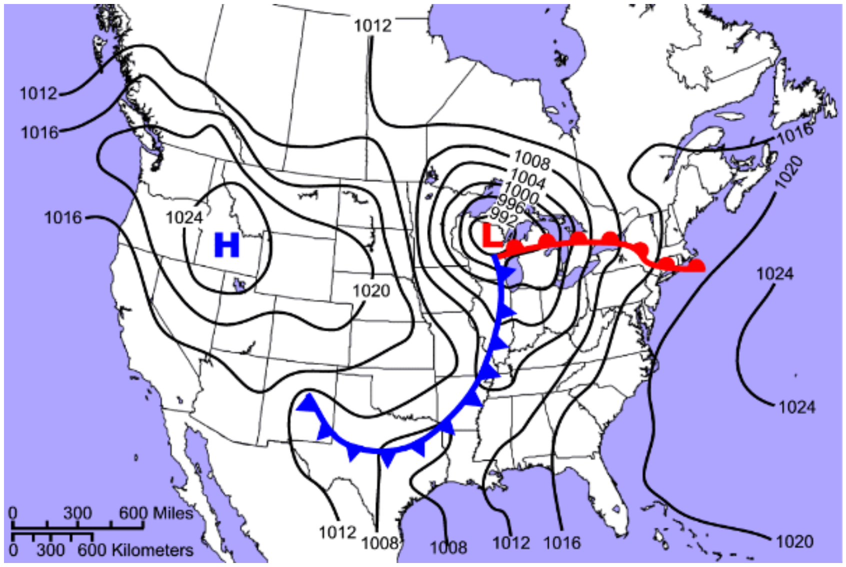


Graph of f in $\mathbb{R}^3$



Level curves of f in $\mathbb{R}^2$

From the South Carolina State Climatology Office
http://www.dnr.sc.gov/climate/sco/Education/wxmap/wxmap.php
Isobars:



For $f : \mathbb{R}^3 \to \mathbb{R}$, the graph of f is a three-dimensional object in $\mathbb{R}^4$, the set of all points $(x, y, z, f(x, y, z))$.

We can get an idea of the shape of the graph of f by drawing some *level surfaces* in $\mathbb{R}^3$, surfaces $f(x, y, z) = k$ for various values of k. If we do this for equally spaced values of k, we get a kind of three-dimensional analogue of a topographical map.

This is pretty much the best we can do to visualize the graph of f, as long as we can't draw (or understand) four-dimensional pictures.

If $f(x, y, z)$ represents some physical quantity (temperature, barometric pressure, ...), the level surfaces of f (isotherms, isobars, ...) are often used to visualize the situation.

Example: $f(x, y, z) = x^2 + y^2 + z^2$. The graph of f is a three-dimensional object sitting in $\mathbb{R}^4$. The level surfaces of f are spheres $x^2 + y^2 + z^2 = k$ of radius $\sqrt{k}$ in $\mathbb{R}^3$.

(The two-dimensional analogue is the function $f(x, y) = x^2 + y^2$ whose graph is the paraboloid in our first set of pictures.)

Example: $f(x, y, z) = x^2 + y^2 - z^2$. The level surfaces of f are hyperboloids — hyperboloids of two sheets for $k < 0$, a double cone for $k = 0$, and hyperboloids of one sheet for $k > 0$.

(The two-dimensional analogue is the function $f(x, y) = x^2 - y^2$ whose graph is the saddle in our second set of pictures.)

Exercise: Draw some level curves of the function $f(x, y) = x^2 + y$. Then try to draw the graph of f. Remember that level curves are in the plane, $\mathbb{R}^2$, but the graph of f is a surface in $\mathbb{R}^3$.

Exercise: Draw some level curves of the function $f(x, y) = 4x^2 + y^2$. Then try to draw the graph of f.

Exercise: Draw some level surfaces $f(x, y, z) = k$ of the function $f(x, y, z) = \dfrac{x+y}{z^2+1}$. Try $k = 1$, $k = 0$, $k = -1$.

Some solutions from last time:

Exercise: Parametrize the intersection of the elliptical cone $z^2 = x^2 + 4y^2$ with the plane $z = 2$.

$$\vec{r}(t) = \langle 2\cos(t), \sin(t), 2\rangle \qquad 0 \le t \le 2\pi$$

Write down an integral representing the arc length of this curve. (Do not try to evaluate this integral.)

$$\int_0^{2\pi} \sqrt{4\sin^2(t) + \cos^2(t)}\, dt = \int_0^{2\pi} \sqrt{3\sin^2(t) + 1}\, dt$$

If an object travels along the curve with position function given by the parametrization you chose, find the tangential and normal components of the object's acceleration, and the curvature of the curve, at the points $(2, 0, 2)$ and $(0, 1, 2)$. Use geometrical reasoning if you can.

(Hint: You're only interested in two points here, so don't do everything in full generality.)

$$\vec{v} = \langle -2\sin(t), \cos(t), 0\rangle \qquad \vec{a} = \langle -2\cos(t), -\sin(t), 0\rangle \qquad \frac{ds}{dt} = \sqrt{4\sin^2(t) + \cos^2(t)}$$

At $(2, 0, 2)$, we have $t = 0$, so

$$\vec{v} = \langle 0, 1, 0\rangle \qquad \vec{a} = \langle -2, 0, 0\rangle = a_{\mathbf{N}}\vec{N} \qquad \frac{ds}{dt} = 1$$

$$a_{\mathbf{T}} = 0 \qquad a_{\mathbf{N}} = |a_{\mathbf{N}}\vec{N}| = 2 \qquad \kappa = 2$$

At $(0, 1, 2)$, we have $t = \frac{\pi}{2}$, so

$$\vec{v} = \langle -2, 0, 0\rangle \qquad \vec{a} = \langle 0, -1, 0\rangle = a_{\mathbf{N}}\vec{N} \qquad \frac{ds}{dt} = 2$$

$$\vec{a}_{\mathbf{T}} = 0 \qquad a_{\mathbf{N}} = |a_{\mathbf{N}}\vec{N}| = 1 \qquad \kappa = \frac{1}{4}$$

Note: At both points we have $\vec{a} \cdot \vec{v} = 0$, which tells us that $\vec{a}$ is perpendicular to $\vec{v}$, so

$$a_{\mathbf{T}}\vec{T} = \vec{0} \quad \text{and} \quad a_{\mathbf{N}}\vec{N} = \vec{a}.$$

We find κ using $a_{\mathbf{N}} = \left(\frac{ds}{dt}\right)^2 \kappa$.