Math 8
Winter 2020
Section 1
January 8, 2020

First, the takeaway from last class:

1. The n^{th} (degree n) Taylor polynomial for $f(x)$ centered at the point a, denoted $T_n(x)$, is the polynomial of degree n that is the best approximation to $f(x)$ near $x = a$. (If $a = 0$, this is called a Maclaurin polynomial.)

2. That is to say, at the point $x = a$, the Taylor polynomial $T_n(x)$ has the same value, first derivative, second derivative, $\ldots n^{th}$ derivative as $f(x)$:

$$T_n(a) = f(a)$$

$$T_n'(a) = f'(a)$$
$$T_n''(a) = f''(a)$$
$$\vdots$$
$$T^{(n)}(a) = f^{(n)}(a)$$

3. The formula is $T_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

4. $T_n(x)$ is sometimes a good approximation to $f(x)$, if n is large enough, but not always.

   How can we tell?

The preliminary homework problems:

1. Find the n^{th} degree Taylor polynomial $T_n(x)$ for the function $f(x) = \dfrac{1}{1-x}$ centered at the point $a = 0$.

   Hint: It may be helpful to think of this as $f(x) = (1-x)^{-1}$.

   For the remaining problems, $T_n(x)$ will mean this polynomial.

   Answer: $\displaystyle\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$.

2. Find $T_n(x)$ for the following values of n and x.

   (a) $x = 1$ and $n = 0, 1, 2, 3$.
   Answer: $1, 2, 3, 4$.

   (b) $x = -1$ and $n = 0, 1, 2, 3$.
   Answer: $1, 0, 1, 0$.

   (c) $x = \frac{1}{2}$ and $n = 0, 1, 2, 3$.
   Answer: $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}$.

3. What does each of the following approach (if anything) as $n \to \infty$? Why? (We haven't discussed limits of this sort yet, so think intuitively.)

   (a) $T_n(1)$.
   Answer: ∞, because $T_n(1) = n + 1$.

   (b) $T_n(-1)$.
   Answer: It does not approach anything; it oscillates between 1 and 0.

   (c) $T_n(\frac{1}{2})$.
   Answer: 2, because each time n increases by 1 the difference from 2 is cut in half, so the difference from 2 approaches 0.

Definition: An infinite sequence is a sequence $(a_1, a_2, a_3, \ldots)$, often denoted $\Big(a_n\Big)_{n=1}^{\infty}$, or sometimes just $\Big(a_n\Big)$. Our textbook uses the alternate notation $\Big\{a_n\Big\}_{n=1}^{\infty}$, with curly brackets. You can use either.

It is not necessary to start at 1; we can have $\Big(a_n\Big)_{n=0}^{\infty}$, or $\Big(a_n\Big)_{n=3}^{\infty}$, etc.

Definition: If L is a number, then

> $\lim\limits_{n\to\infty} a_n = L$ means for every $\varepsilon > 0$, there is an N such that for all $n > N$, we have $|L - a_n| < \varepsilon$.

Here ϵ is how close to L you want to make a_n, and N is how far out in the sequence you have to get to always be that close. Then $n > N$ means you are at least that far out in the sequence, and $|L - a_n| < \varepsilon$ means a_n is that close to L.

We can take this same approach to sequences approaching infinity, interpreting "close to infinity" as "big."

Definition:

> $\lim\limits_{n\to\infty} a_n = \infty$ means for every M, there is an N such that for all $n > N$, we have $a_n > M$.
>
> $\lim\limits_{n\to\infty} a_n = -\infty$ means for every M, there is an N such that for all $n > N$, we have $a_n < M$.

Definition: If $\lim\limits_{n\to\infty} a_n = L$ and L is a number, then the sequence $\Big(a_n\Big)$ converges to L, and is a convergent sequence.

Otherwise, it diverges, and is a divergent sequence. If $\lim\limits_{n\to\infty} a_n = \infty$ or $\lim\limits_{n\to\infty} a_n = -\infty$ we say the sequence diverges to infinity or to minus infinity. We never say a sequence converges to infinity.

Example: Use the definition of limit to show

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+2} = 1.$$

Example: Show

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } |x| < 1; \\ 1 & \text{if } x = 1; \\ \text{undefined} & \text{otherwise .} \end{cases}$$

You don't need to use the formal definition of limit.

Example: Show

$$\lim_{n\to\infty}\left(\frac{2^n}{n!}\right) = 0.$$

You don't need to use the formal definition of limit.

Note:

$$\frac{2^n}{n!} = \frac{2\cdot 2\cdot 2\cdots 2}{1\cdot 2\cdot 3\cdots n} = \left(\frac{2}{1}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right)\left(\frac{2}{4}\right)\left(\frac{2}{5}\right)\cdots\left(\frac{2}{n}\right).$$

Note: We are particularly interested in sequences of Taylor polynomials evaluated at some particular x, sequences of the form $\Big(T_n(x)\Big)_{n=0}^{\infty}$. If T_n is the degree n Taylor polynomial (which we sometimes call the n^{th} Taylor polynomial) for f centered at a, we hope that $\lim_{n\to\infty} T_n(x) = f(x)$.

Example: In the preliminary homework, you saw that the n^{th} Taylor polynomial for $f(x) = \dfrac{1}{1-x}$ centered at 0 is

$$T_n(x) = 1 + x + x^2 + \cdots + x^n.$$

We can show

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}.$$

We can rewrite this as

$$\frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

so

$$T_n(x) = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \left(\frac{1}{1-x} - \frac{x^{n+1}}{1-x}\right)$$

Whenever $|x| < 1$ we have

$$\lim_{n\to\infty} T_n(x) = \frac{1}{1-x}.$$

If $|x| \geq 1$ then the sequence $(T_n(x))$ diverges.

Example: As the demo we saw last class suggested, if T_n is the n^{th} Maclaurin polynomial for $\sin x$ centered at 0, then for every x we have $\lim_{n\to\infty} T_n(x) = \sin x$. We will not have the tools to prove this in Math 8, but it is a fact.

In the textbook (Chapter 11, Section 1) or the notes you can find a lot of theorems about limits and convergence of sequences. For example, there are arithmetic rules, such as

$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$

Most of these rules make good intuitive sense, and are pretty much the same as rules you learned for limits of functions. You do not need to memorize these rules or know their names, but you are free to use them.

The only exception is that if a problem asks you to show convergence using the definition of limit, you must use the definition of limit, not these rules.

Here are a couple that are worth noting:

Squeeze Theorem: If $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = a$, and $a_n \le c_n \le b_n$ for all n (or for all large enough n), then $\lim_{n\to\infty} c_n = a$.

Definition: A sequence $\left(a_n\right)$ is increasing if $a_n \le a_{n+1}$ for all n, decreasing if $a_n \ge a_{n+1}$ for all n, and monotonic if it is either increasing or decreasing.

Monotonic Sequence Theorem: An increasing sequence must either converge to a limit or diverge to infinity. A decreasing sequence must either converge to a limit or diverge to minus infinity.

Exercise: Find $\lim_{n\to\infty}\left(\dfrac{n^3-5}{4n^3+2}\right)$.

Hint: Apply l'Hôpital's rule to the function $f(x) = \dfrac{x^3-5}{4x^3+2}$.

Exercise: Argue that if c is any constant, then $\lim_{n\to\infty}\left(\frac{c^n}{n!}\right) = 0$.
Argue that $\lim_{n\to\infty}\left(\frac{n^n}{n!}\right) = \infty$.
You do not need to use the formal definition of limit.

Exercise: Use the definition of limit to show that $\lim_{n\to\infty}\left(\frac{n^2}{n^2+1}\right) = 1$.
In other words: Given $\varepsilon > 0$, find N large enough so that

$$n > N \implies \left|\left(\frac{n^2}{n^2+1}\right) - 1\right| < \varepsilon.$$

Hint: $1 = \left(\frac{n^2+1}{n^2+1}\right)$.