Math 8
Winter 2020
Section 1
February 19, 2020

First, some important points from the last class:

For $f : \mathbb{R}^2 \to \mathbb{R}$, the graph of f is a surface in $\mathbb{R}^3$, the set of all points $(x, y, f(x, y))$.

The level curves of f are curves $f(x, y) = k$ in $\mathbb{R}^2$. We can think of them as projections onto the xy-plane of horizontal slices of the graph of f.

We can draw a contour plot of f by drawing level curves $f(x, y) = k$ for equally spaced values of k. The contour plot is in $\mathbb{R}^2$ and is like a topographical map of the graph of f.

By looking at the contour plot we can see where the graph of f is steepest, what direction the surface slopes in, and where high and low points are.

For $f : \mathbb{R}^3 \to \mathbb{R}$, the graph of f is in $\mathbb{R}^4$, the set of all points $(x, y, z, f(x, y, z))$. We cannot draw it.

The level surfaces of f are surfaces $f(x, y, z) = k$ in $\mathbb{R}^3$. We can draw them.

If $f(x, y, z)$ is the temperature at (x, y, z), the level surfaces of f are isotherms. If f gives barometric pressure, the level surfaces are isobars.

For $f : \mathbb{R}^n \to \mathbb{R}$, the graph of f is in $\mathbb{R}^{n+1}$, the set of all points $(x, y, z, \ldots, f(x, y, z, \ldots))$. The level sets of f are in $\mathbb{R}^n$. They have equations $f(x, y, z, \ldots) = k$.

Level curves and level surfaces are two kinds of level sets.

A function of several variables is a function

$$f : \mathbb{R}^n \to \mathbb{R}.$$

Today we are going to talk about limits of functions of several variables.

Our definition of limit always has the same form: For every desired output accuracy, there is a required input accuracy, such that whenever the input is within the input accuracy, then the output is within the output accuracy.

For functions $f : \mathbb{R}^3 \to \mathbb{R}$ it becomes:

Definition:

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

means for every $\varepsilon > 0$ [desired output accuracy] there is a $\delta > 0$ [required input accuracy] such that, for every (x,y,z),

$$\Bigg[\underbrace{\overbrace{|(x,y,z)-(x_0,y_0,z_0)|}^{\text{distance}} < \delta \;\;\&\;\; (x,y,z) \neq (x_0,y_0,z_0)}_{\text{within input accuracy}}\Bigg] \implies \underbrace{|f(x,y,z)-L| < \varepsilon}_{\text{within output accuracy}}.$$

Definition: The function $f(x,y,z)$ is continuous at (x_0,y_0,z_0) if

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0).$$

The definitions for $f : \mathbb{R}^2 \to \mathbb{R}$, and for $f : \mathbb{R}^n \to \mathbb{R}$, are similar.

If $F : \mathbb{R}^m \to \mathbb{R}^n$, we take limits coordinatewise. So if

$$F(x,y) = (F_1(x,y), F_2(x,y)),$$

then

$$\lim_{(x,y)\to(x_0,y_0)} F(x,y) = \left(\lim_{(x,y)\to(x_0,y_0)} F_1(x,y), \lim_{(x,y)\to(x_0,y_0)} F_2(x,y)\right).$$

Warning: For $f : \mathbb{R} \to \mathbb{R}$, there are two ways x can approach a, from the right and from the left. For $f : \mathbb{R}^n \to R$, there are infinitely many ways $(x_1, \ldots, x_n)$ can approach $(a_1, \ldots, a_n)$: along lines, along parabolas, along spirals…

To show a limit does not exist, we can show two different ways of approach that lead to different limits. (This is like showing the right-hand and left-hand limits are unequal.) To show a limit equals L, you need to show the limit will be L on every possible approach. It is not enough to check some example approaches.

The theorems that should be true about the limits of sums, products, compositions (paying attention to continuity), etc., and the squeeze theorem, are true, and you can use them.

Examples:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}:$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 + y^2}:$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}:$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}:$$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}:$$

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{r}{\theta}$ where (r,θ) are polar coordinates of (x,y), and we choose $r \geq 0$ and $0 < \theta \leq 2\pi$.

Exercises: (hints available)

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} \big((x^2 + y^2)\ln(x^2 + y^2)\big)$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$$

Exercise: (from last class) Draw some level surfaces $f(x, y, z) = k$ of the function $f(x, y, z) = \dfrac{x+y}{z^2+1}$. Try $k = 1$, $k = 0$, $k = -1$.

Solution: Let's look at $k = -1$. Our equation becomes

$$\frac{x+y}{z^2+1} = -1 \qquad z = \pm\sqrt{-(x+y)-1}.$$

This surface has a top part ($z \geq 0$) and a bottom part ($z \leq 0$) that are reflections of each other across the xy-plane, so we can start by looking at the top half

$$z = \sqrt{-(x+y)-1}.$$

This is a function, and to sketch its graph we can start by noticing that level curves $z = c$ are parallel lines $x + y = -(c^2 + 1)$. The lowest part of the surface is where $z = 0$, the line $x + y = -1$ in the xy-plane. The intersection with the plane $x = 0$ is the $z \geq 0$ portion of the parabola $y = -z^2 - 1$. The surface looks as if a vertical half-plane anchored on the line $x + y = -1$ were bent over so its cross-section formed half of a sideways-facing parabola.

The bottom half of the surface is the mirror image.

In the pictures below, this level surface is in green (the leftmost surface), the level surface $k = 0$ is in red, and the level surface $k = 1$ is in yellow (the rightmost surface). (The points come from the fact that the surfaces are cut off by a cube.) In the middle picture the direction of view is horizontal, and parallel to the plane $x + y = 0$ (the level surface for $k = 0$), and in the right hand picture the view is from the top down.