Math 8
Fall 2019
Section 2
November 4, 2019

First, some important points from the last class:

Definition:

$$\lim_{(x,y,z)\to(x_0,y_0,z_0} f(x,y,z) = L$$

means for every $\varepsilon > 0$ [desired output accuracy] there is a $\delta > 0$ [required input accuracy] such that, for every (x,y,z),

$$\Big[\underbrace{\overbrace{|(x,y,z)-(x_0,y_0,z_0)|}^{\text{distance between } (x,y,z) \text{ and } (x_0,y_0,z_0)} < \delta \ \& \ (x,y,z) \neq (x_0,y_0,z_0)}_{\text{within input accuracy}}\Big] \implies \underbrace{|f(x,y,z)-L| < \varepsilon}_{\text{within output accuracy}}.$$

Definition: The function $f(x,y,z)$ is continuous at (x_0,y_0,z_0) if

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0).$$

The definitions for $f:\mathbb{R}^2\to\mathbb{R}$, and for $f:\mathbb{R}^n\to\mathbb{R}$, are similar.

If $F:\mathbb{R}^m\to\mathbb{R}^n$, we take limits coordinatewise. So if

$$F(x,y) = (F_1(x,y), F_2(x,y)),$$

then

$$\lim_{(x,y)\to(x_0,y_0)} F(x,y) = \left(\lim_{(x,y)\to(x_0,y_0)} F_1(x,y), \lim_{(x,y)\to(x_0,y_0)} F_2(x,y)\right).$$

To show a limit does not exist, we can show two different ways of approach that lead to different limits. (This is like showing the right-hand and left-hand limits are unequal.)

To show a limit exists (and equals L), it is not enough to check different approaches. Some tools you can use to show limits exist:

Breaking up an expression as a sum, product, composition...

The squeeze theorem.

Polar coordinates for limits as $(x,y)\to(0,0)$.

L'Hôpital's rule, but be warned that this is *only* if you have already reduced the problem to the limit of a function of one variable. This is an important warning. We do not have a two-dimensional version of L'Hôpital's rule.

Preliminary homework:

Let $f(x, y) = x^2 + y^2$, let γ_1 be the intersection of the graph of f with the plane $x = 2$, and let γ_2 be the intersection of the graph of f with the plane $y = 1$.

(1.) Let $h(x) = f(x, 1)$. Find $h'(2)$. What does this number say about the curve formed by intersecting the graph of f with the plane $y = 1$?

$h(x) = f(x, 1) = x^2 + 1$ so $h'(x) = 2x$ and $h'(2) = 4$. This is the slope (considering the z-axis to be vertical) of the curve at that point (where $y = 1$ and $x = 2$, or $(2, 1, 5)$).

You could also call it the slope of the surface in the x-direction at $(2, 1, 5)$. It is $\frac{d}{dx}(f(x, 1))$ evaluated at $x = 2$.

(2.) Let $g(y) = f(2, y)$. Find $g'(1)$. What does this number say about the curve formed by intersecting the graph of f with the plane $x = 2$?

$g(y) = f(2, y) = 4 + y^2$ so $g'(y) = 2y$ and $g'(1) = 2$. This is the slope (considering the z-axis to be vertical) of the curve at that point (where $x = 2$ and $y = 1$, or $(2, 1, 5)$).

You could also call it the slope of the surface in the y-direction at $(2, 1, 5)$. It is $\frac{d}{dy}(f(2, y))$ evaluated at $y = 1$.

(3.) Let $a = h'(2)$ and $b = g'(1)$. Find numbers c and d such that the plane $z = ax + by + d$ and the graph of f both contain the point $(2, 1, c)$.

$c = f(2, 1) = 5$.

We need $5 = a(2) + b(1) + d = 4(2) + 2(1) + d = 10 + d$ so $d = -5$

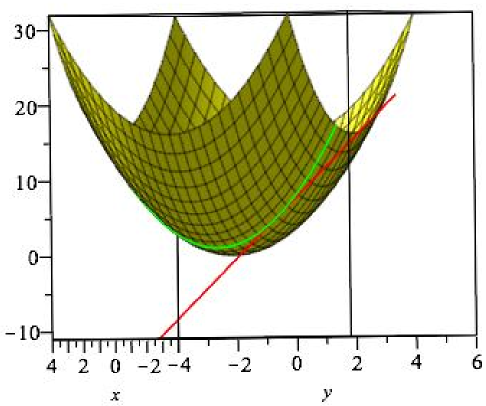
Note: Why $a = h'(2)$ and $b = g'(1)$? These are the numbers we said we could think of as the slopes of the surface in the x- and y-directions.

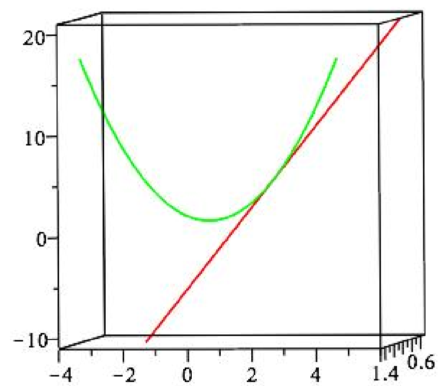
Today: Partial Derivatives

Example: Consider the surface S that is the graph of $f(x, y) = x^2 + y^2$.
How can we describe the slope (treating the z-axis as vertical) of S?
As an example, consider the point $(2, 1, 5)$ on S.
(This is the same surface and point as in the preliminary homework.)

If we slice S in the plane $x = 2$, we get a parabola, $z = 4 + y^2$, and we can compute the rate of change of z with respect to y when $y = 1$,

$$\left.\frac{dz}{dy}\right|_{y=1} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=1} = \left.(2y)\right|_{y=1} = 2$$





If we slice S in the plane $y = 1$, we get a parabola, $z = x^2 + 1$, and we can compute the rate of change of z with respect to x when $x = 2$,

$$\left.\frac{dz}{dx}\right|_{x=2} = \left.\frac{d}{dx}(x^2 + 4)\right|_{x=2} = \left.(2x)\right|_{x=2} = 4$$

Geometrically, these are the slopes (vertical rise over horizontal run, treating the z-axis as vertical) of the tangent lines to S at $(2, 1, 5)$ in the planes $y = 1$ and $x = 2$.

These are the *partial derivatives* of $f(x, y)$ with respect to x and with respect to y.

Definition: The partial derivative of $f(x,y)$ with respect to x at the point (x_0, y_0) is the derivative of the function of x we get by setting y to have constant value y_0:

$$\underbrace{\frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0) = D_x f(x_0,y_0)}_{\text{Three different notations for partial derivative.}}$$

$$= \frac{d}{dx}\left(f(x,y_0)\right)\Big|_{x=x_0}.$$

Example: The partial derivatives of $f(x,y) = x^2 - y^2$, computed by treating the other variable as a constant, are

$$\frac{\partial f}{\partial x}(x,y) = \qquad \frac{\partial f}{\partial y}(x,y) =$$

$$f_x(3,1) = \qquad f_y(3,1) =$$

If $z = f(x,y)$ we may also call the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Example: If $z = y\sin(xy)$ then

$$\frac{\partial z}{\partial x} = \qquad \frac{\partial z}{\partial y} =$$

Definition: The partial derivative of $f(x,y,z)$ with respect to x at the point $(x_0, y_0, z_0))$ is the derivative of the function of x we get by setting y and z to have constant values y_0 and z_0:

$$\frac{\partial f}{\partial x}(x_0,y_0,z_0) = f_x(x_0,y_0,z_0) = D_x f(x_0,y_0,z_0) = \frac{d}{dx}\left(f(x,y_0,z_0)\right)\Big|_{x=x_0}.$$

Example: The partial derivatives of $f(x,y,z) = xyz - z$, computed by treating the other variables as a constant, are

$$\frac{\partial f}{\partial x}(x,y,z) = \qquad \frac{\partial f}{\partial y}(x,y,z) = \qquad \frac{\partial f}{\partial z}(x,y,z) =$$

The partial derivatives of f are themselves functions from $\mathbb{R}^2 \to \mathbb{R}$, and we can take their partial derivatives, called the second partial derivatives of f.

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Notice the order of x and y in the different notations.

Example: Find all the first and second partial derivatives.

$$f(x, y) = x^2 + 2xy - y^2$$

$$f_x(x, y) =$$

$$f_{xx}(x, y) = \qquad \boxed{f_{xy}(x, y) = \qquad\qquad}$$

$$f_y(x, y) =$$

$$f_{yy}(y, x) = \qquad \boxed{f_{yx}(x, y) = \qquad\qquad}$$

$$f(x, y) = e^x \sin(xy)$$

$$f_x(x, y) =$$

$$f_{xx}(x, y) =$$

$$\boxed{f_{xy}(x, y) = \qquad\qquad\qquad\qquad\qquad\qquad}$$

$$f_y(x, y) =$$

$$f_{yy}(x, y) =$$

$$\boxed{f_{yx}(x, y) = \qquad\qquad\qquad\qquad\qquad\qquad}$$

Notice something?

Theorem (Clairaut's theorem): If suitable hypotheses hold (the first and second partial derivatives of f are continuous near the point in question), the corresponding mixed second partial derivatives of a function are always equal. That is,

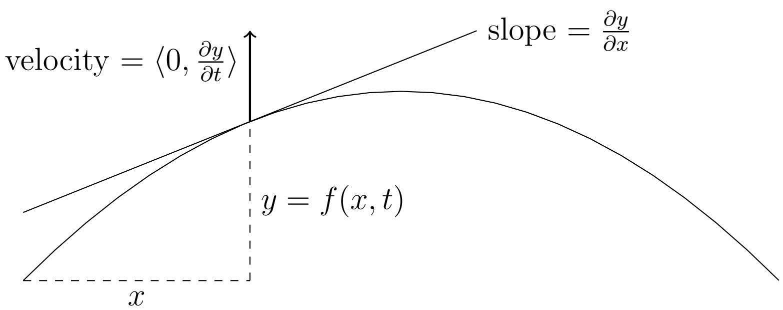
$$f_{xy} = f_{yx} \qquad f_{xz} = f_{zx} \qquad f_{yz} = f_{zy}$$

Example: The motion of a vibrating string, anchored on the x-axis at points $x = a$ and $x = b$ and vibrating in the xy-plane, may be described by a function $f(x, t)$ giving the y-coordinate at time t of the point on the string with x-coordinate equal to x.

At a particular time t and point on the string x, the physical significance of the first and second partial derivatives is:

$f_x(x, t)$ is the slope of the string at point x.

$f_t(x, t)$ is the vertical component of the velocity of point x on the string.



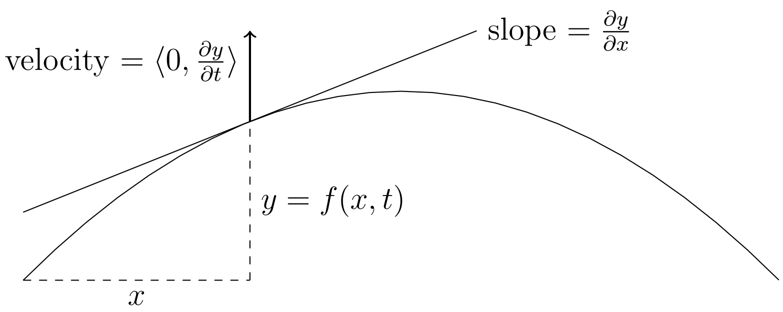
Snapshot at time t.

$f_{xx}(x, t) = \dfrac{\partial}{\partial x}(f_x(x, t))$ is the second derivative of the y-coordinate of the string (this determines the curvature).

$f_{xt}(x, t) = \dfrac{\partial}{\partial t}(f_x(x, t))$ is the rate at which the slope of the string is changing over time.

$f_{tx}(x, t) = \dfrac{\partial}{\partial x}(f_t(x, t))$ is the rate at which the vertical component of the instantaneous velocity, at a fixed time, changes with respect to distance along the string.

$f_{tt}(x, t) = \dfrac{\partial}{\partial t}(f_t(x, t))$ is the vertical component of acceleration of point x on the string.



Snapshot at time t.

Why does it make sense that $f_{xt} = f_{tx}$?

$f_{xt}(x,t)$ is the rate at which the slope of the string is changing over time.

$f_{tx}(x,t)$ is the rate at which the vertical component of the instantaneous velocity, at a fixed time, changes with respect to distance along the string.

If f_{tx} is positive, the point at $x+\Delta x$ (for $\Delta x > 0$) is moving upward faster than the point at x, so the slope is increasing and f_{xt} is positive. The larger f_{tx} is, the more the speeds differ, the faster the slope increases, the larger f_{xt} is.

Why does it make sense that in this physical situation f satisfies the wave equation,

$$f_{tt} = c^2 f_{xx} \quad \text{or} \quad \frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

for some constant c?

$f_{xx}(x,t)$ is the second derivative of the y-coordinate of the string (this determines the curvature).

$f_{tt}(x,t)$ is the vertical component of acceleration of point x on the string.

When f_{xx} is large and negative, the curvature of the string is large and the string is concave down (as in the picture), so the tension exerts large downward force, for large downward (negative) acceleration.

Example: Show the function $f(x,t) = \cos(3x+2t)$ satisfies the wave equation, and find the value of c.

$$f_x(x,t) = -3\sin(3x+2t) \qquad f_{xx}(x,t) = -9\cos(3x+2t)$$

$$f_t(x,t) = -2\sin(3x+2t) \qquad f_{tt}(x,t) = -4\cos(3x+2t) = \frac{4}{9}\left(-9\cos(3x+2t)\right)$$

$$f_{tt}(x,t) = \left(\frac{2}{3}\right)^2 f_{xx}(x,t)$$

$$c = \frac{2}{3}$$

We can use implicit differentiation to find partial derivatives.

Example: Find the slope (treating the z-axis as vertical) of the tangent line to the sphere $x^2 + y^2 + z^2 = 50$ at the point $(3, 4, 5)$ that lies in the plane $x = 3$.

Note that we are looking for $\dfrac{\partial z}{\partial y}$ at this point, so we are thinking of x as a constant, and z as a function of y.

Method 1: Write z as a function of x and y, and then find the partial derivative:

$$z = \pm\sqrt{50 - x^2 - y^2} = \sqrt{50 - x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(50 - x^2 - y^2)^{-\frac{1}{2}}(-2y) = -y(50 - x^2 - y^2)^{-\frac{1}{2}}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(3,4)} = -4(50 - 9 - 16)^{-\frac{1}{2}} = \frac{-4}{5}.$$

Method 2: Implicitly differentiate the equation with respect to y. We are taking partial derivatives with respect to y, treating x as a constant, and z as a function of y.

$$\begin{aligned} x^2 + y^2 + z^2 &= 50 \\ 0 + 2y + 2z\frac{\partial z}{\partial y} &= 0 \\ \frac{\partial z}{\partial y} &= \frac{-y}{z} \\ \left.\frac{\partial z}{\partial y}\right|_{(x,y,z)=(3,4,5)} &= \frac{-4}{5}. \end{aligned}$$

Example: Find an equation for this tangent line.

This line lies in the plane $x = 3$, goes through the point $(3, 4, 5)$, and has slope $\dfrac{-4}{5}$, so if y changes by 1, then z changes by $\dfrac{-4}{5}$ (and x does not change at all). This tells us the vector $\left\langle 0, 1, \dfrac{-4}{5} \right\rangle$ gives the direction of the line, so an equation is

$$\langle x, y, z\rangle = \langle 3, 4, 5\rangle + t\left\langle 0, 1, \frac{-4}{5}\right\rangle.$$

Example: If you look up Brewer-Dobson circulation on Wikipedia, you will find an article illustrated with a contour plot. The function being plotted here is ozone concentration (measured in Dobson units, DU) as a function of latitude (measured in degrees) and height (measured in kilometers). We will use x, y, and z to denote latitude, height, and ozone concentration.

Look at the point on the graph representing latitude -30 deg. and height 27 km. There is a 14 on the plot, indicating a contour line representing ozone concentration of 14 DU. This means, of course, that at latitude -30 degrees and height 27 kilometers, the ozone concentration is 14 DU.

If we move upward on the plot from that point, representing increasing height (y), the numbers on the contour lines decrease, meaning that ozone concentration (z) is decreasing. This means that at that point, the partial derivative $\frac{\partial z}{\partial y}$ is negative. We can estimate the value of the partial derivative from the contour plot. The vertical distance at that point between the contour lines labeled 14 and 12 appears to be about the same as the distance between heights 26 and 28 in the scale at the left. This means that an increase in height of 2 kilometers produces a decrease in ozone level of 2 DU, so $\frac{\partial z}{\partial y} \approx -\frac{2}{2} = -1$. The units of $\frac{\partial z}{\partial y}$ are DU per kilometer.

Try estimating the value of $\frac{\partial z}{\partial x}$ at that point. It appears to be positive, and to have value about $\frac{1}{5}$ DU per degree. That means that increasing latitude (moving north) by one degree produces an increase in ozone level of $\frac{1}{5}$.

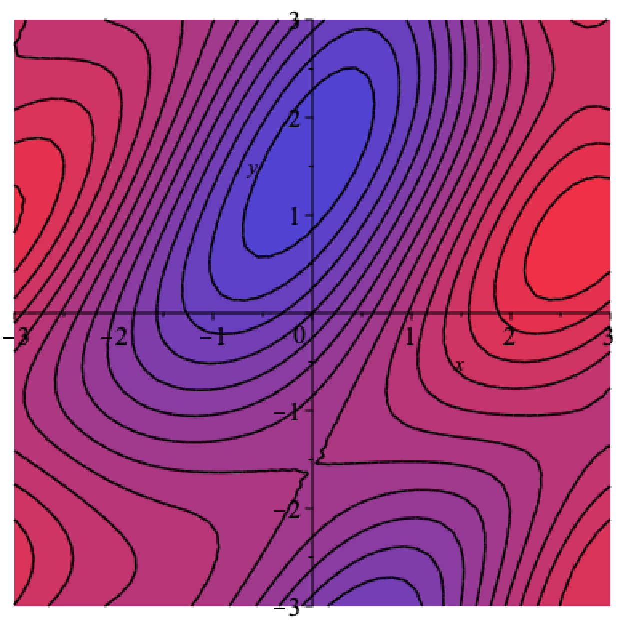
Can you find a location where $\frac{\partial z}{\partial x}$ is negative and $\frac{\partial z}{\partial y}$ is positive? What does this say about ozone distribution in that region?

Exercise: In this contour plot, blue represents higher values of $f(x,y)$, and the contour lines are spaced .25 apart (so if (a,b) is on one contour line and (c,d) is on the next, then $f(a,b)$ and $f(c,d)$ differ by .25).

Estimate the partial derivatives of f at $(1,1)$ and at $(1,-2)$.

Where on the graph does f_x appear to be largest?

Explain why f_x and f_y both appear to have value 0 at the points $(0,1.5)$ and $(0,-1.5)$. Does the graph of f near $(0,1.5)$ look similar to the graph of f near $(0,-1.5)$, or are those regions of the graph essentially different?



Exercise: The surface S has equation $z = x^2y - y^2x$. Find the line that lies in the plane $y = 2$ and is tangent to S at the point $(1, 2, -2)$.

Exercise: Show that the function

$$f(x, y) = 5e^{3x+1}\sin(3y - 4)$$

satisfies Laplace's equation,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Exercise: We saw that the direction of the line in the plane $x = 3$ tangent to the sphere $x^2 + y^2 + z^2 = 50$ at the point $(3, 4, 5)$ is given by the vector $\left\langle 0, 1, \frac{-4}{5} \right\rangle$.

Find a vector giving the direction of the line in the plane $y = 4$ tangent to the sphere $x^2 + y^2 + z^2 = 50$ at the point $(3, 4, 5)$.

Use this information to find an equation for the plane tangent to the sphere $x^2+y^2+z^2 = 50$ at the point $(3, 4, 5)$.