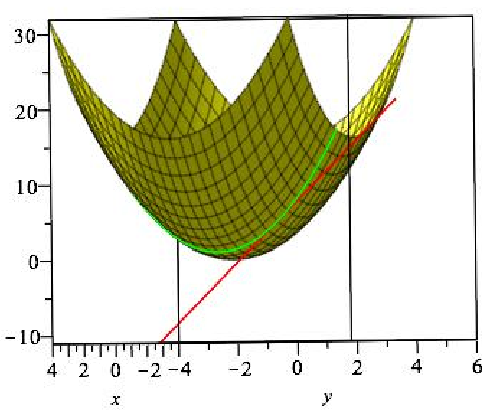
Math 8
Winter 2020
Section 1
February 24, 2020

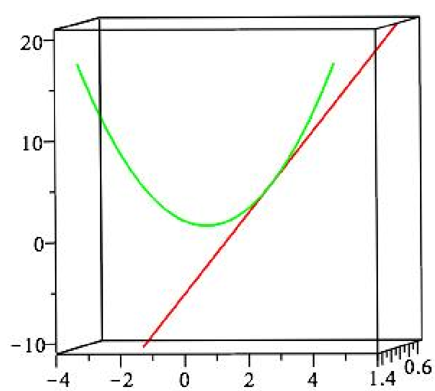
First, some important points from the last class:

Definition: The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is the derivative of the function of x we get by setting y to have constant value y_0:

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = D_x f(x_0, y_0) = \frac{d}{dx}\left(f(x, y_0)\right)\Big|_{x=x_0} =$$
$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

Geometrically, this is the slope (vertical rise over horizontal run, treating the z-axis as vertical) of the tangent line to the graph of f at $(x_0, y_0, f(x_0, y_0)))$ in the plane $y = y_0$.





The second partial derivatives of f include

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

Theorem (Clairaut's theorem): If suitable hypotheses hold, the corresponding mixed second partial derivatives of a function are always equal. That is,

$$f_{xy} = f_{yx} \qquad f_{xz} = f_{zx} \qquad f_{yz} = f_{zy}$$

Example: Find an equation for the tangent plane to the graph of the function

$$f(x,y) = x^2y^2$$

at the point $(1,3,9)$.

The partial derivatives of f at that point are

$$\frac{\partial f}{\partial x}(1,3) = (2xy^2)\Big|_{(x,y)=(1,3)} = 18$$

$$\frac{\partial f}{\partial y}(1,3) = (2x^2y)\Big|_{(x,y)=(1,3)} = 6$$

Vectors in the direction of the lines tangent to the graph of f at that point in vertical planes:

$$x = 1: \qquad \left\langle 0, 1, \frac{\partial f}{\partial y}(1,3) \right\rangle = \langle 0,1,6\rangle$$

$$y = 3: \qquad \left\langle 1, 0, \frac{\partial f}{\partial x}(1,3) \right\rangle = \langle 1,0,18\rangle$$

Vector normal to both tangent lines:

$$\langle 0,1,6\rangle \times \langle 1,0,18\rangle = \langle 18, 6, -1\rangle$$

Equation of plane containing both tangent lines (containing point $(1,3,9)$ and normal to the vector $\langle 18,6,-1\rangle$):

$$\langle x-1, y-3, z-9\rangle \cdot \langle 18,6,-1\rangle = 0$$

$$18(x-1) + 6(y-3) - (z-9) = 0$$

$$\underbrace{(z-9)}_{\Delta z} = \underbrace{18}_{\frac{\partial z}{\partial x}}\underbrace{(x-1)}_{\Delta x} + \underbrace{6}_{\frac{\partial z}{\partial y}}\underbrace{(y-3)}_{\Delta y}$$

$$z = 18(x-1) + 6(y-3) + 9$$

$$z = \left(\frac{\partial f}{\partial x}(1,3)\right)\underbrace{(x-1)}_{\Delta x} + \left(\frac{\partial f}{\partial y}(1,3)\right)\underbrace{(y-3)}_{\Delta y} + f(1,3)$$

Theorem: If the graph of f has a tangent plane at the point $(x_0, y_0, f(x_0,y_0))$, its equation is

$$z = \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(x-x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(y-y_0) + f(x_0,y_0).$$

Theorem: (the same theorem rephrased) If the graph of f has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$, it is the graph of the function

$$L(x,y) = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) + f(x_0, y_0).$$

Definition: The function

$$L(x,y) = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) + f(x_0, y_0)$$

is called the *linearization* of f at (x_0, y_0). We may also call it a linear approximation or a tangent approximation.

For (x, y) near (x_0, y_0), we have $f(x, y) \approx L(x, y)$. Setting $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$ (so $x - x_0 = \Delta x$ and $y - y_0 = \Delta y$), when Δx and Δy are small, we can write

$$f(x,y) \approx L(x,y) = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) + f(x_0, y_0)$$

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(\Delta x) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(\Delta y) + f(x_0, y_0).$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \approx \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(\Delta x) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(\Delta y);$$

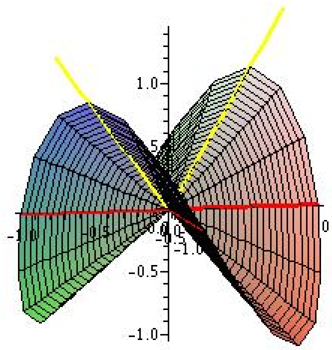
$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.$$

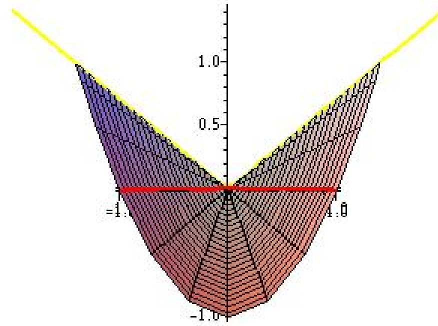
The differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, \quad \text{or} \quad dz = \underbrace{\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}_{\text{One piece for each input variable!}}.$$

Everything works the same for $f : \mathbb{R}^3 \to \mathbb{R}$, or, for that matter, for $f : \mathbb{R}^n \to \mathbb{R}$.

Warning: The fact that f has partial derivatives at a point is *not enough* to guarantee that its graph has a tangent plane there. Here are two pictures of the graph of the function





$f(x,y) = \dfrac{2xy}{\sqrt{x^2+y^2}}$ (setting $f(0,0) = 0$, to make f defined and continuous everywhere). The red lines are the intersections of the graph of f with the planes $x = 0$ and $y = 0$. Both are horizontal, so $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0$. The yellow $\vee$ is the intersection of the graph of f with the plane $x = y$. It is pointed at the origin, and does not have a tangent line there, so the graph of f has no tangent plane at $(0,0)$.

We do, however, have this useful theorem:

Theorem: If the partial derivatives of $f(x,y)$ are defined near (x_0, y_0) and continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Defined near (x_0, y_0) means there is some (possibly tiny) disc with center (x_0, y_0) such that the partial derivatives are defined at all points inside the disc. If this disc has radius δ, it may be called the δ-neighborhood of (x_0, y_0). Some books, instead of defined near (x_0, y_0), may say defined in some neighborhood of (x_0, y_0).

You can compute the partial derivatives of the function f pictured above, and check that although they are defined everywhere, they are not continuous at $(0,0)$.

Example: Show that

$$f(x, y, z) = xyz$$

is differentiable at the point $(1, 2, 1)$, and then use the linear approximation to f to approximate the product of the three numbers 1.01, 1.98, and .99.

The partial derivatives of f are

$$\frac{\partial f}{\partial x}(x, y, z) = \qquad \frac{\partial f}{\partial y}(x, y, z) = \qquad \frac{\partial f}{\partial z}(x, y, z) =$$

They are defined and continuous everywhere (because they are polynomials), so by the theorem, f is differentiable everywhere.

For small values of Δx, Δy, and Δz, we can say

$$f(1 + \Delta x, 2 + \Delta y, 1 + \Delta z) \approx$$

Example: Find an equation for the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 169$$

at the point $(3, 4, 12)$.

Example: Use differentials to approximate the volume of metal in a cylindrical can of height 6 inches and radius 2 inches, if the top and bottom of the can are .006 inches thick, and the curved sides are .004 inches thick.

We can express volume as a function of height and radius as

$$V = \pi r^2 h.$$

We want the difference in volume between the inside and outside of the can, so we are looking for ΔV when $\Delta r = .004$ and $\Delta h = .012$, given $r \approx 2$ and $h \approx 6$. We have

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$dV = (2\pi r h)\, dr + \left(\pi r^2\right) dh$$

$$\Delta V \approx (2\pi r h)\, \Delta r + \left(\pi r^2\right) \Delta h.$$

Plugging everything in:

$$\Delta V \approx (2\pi(2)(6))\,(.004) + \left(2\pi(2)^2\right)(.012) = (.192)\pi$$

The can contains approximately $.192\pi$ cubic inches of metal.

Exercise: The temperature at point (x, y, z) (where distances are in meters) is given by the function $f(x, y, z) = x^2 + 2y^2 + z^4$ (in degrees Celsius).

What are the units of $\dfrac{\partial f}{\partial x}$? $\boxed{}$

Find the linearization of the function f at the point $(1, 1, 1)$.

The differential df of the function f is given by

$$df = \boxed{}$$

Suppose that points P and Q are both near the point $(1, 1, 1)$, and the displacement from P to Q is $\overrightarrow{PQ} = \langle .01, .02, -.02\rangle$. Use differentials to approximate the change in temperature when moving from point P to point Q.

Exercise: Use implicit differentiation to find the partial derivatives of z with respect to x and with respect to y on the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 3$$

at the point $(3, 4, 5)$. Then find an equation for the tangent plane to the ellipsoid at that point.

Use the linear approximation to approximate the z-coordinate of a point on the ellipsoid whose x- and y-coordinates are 3.02 and 4.01.

Exercise: Show that any function of the form

$$f(x, y) = ae^{bx} \sin(by),$$

where a and b are constants, satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Exercise: Check directly that Clairaut's Theorem holds of any function of the form

$$f(x, y) = g(x)h(y),$$

where g and h are differentiable functions. (Hint: If x is constant, then $g(x)$ is also constant.)