Math 8
Fall 2019
Section 2
November 8, 2019

First, some important points from the last class:

Theorem: If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function whose graph has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ (in other words, f is differentiable at (x_0, y_0)), then the tangent plane is the graph of the function

$$L(x,y) = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) + f(x_0, y_0).$$

Theorem: If the partial derivatives of $f(x,y)$ are defined near (x_0, y_0) and continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

When f is differentiable at (x_0, y_0), we can approximate $f(x,y)$ near (x_0, y_0) by

$$f(x,y) \approx L(x,y).$$

This is called the *linear approximation* or *tangent approximation* to f near (x_0, y_0). The function $L(x,y)$ is called the *linearization* of f at (x_0, y_0).

Definition: The differential of f is defined by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

When f is differentiable, we can use the differential for making approximations:

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

In the preliminary homework, we had that θ is a function of x and y, and x and y are functions of time t, and we wanted to find $\dfrac{d\theta}{dt}$ at a particular time t_0. Probably you solved for θ as a function of t and then took the derivative. Here is a different way to think about it:

Near the point $x = x_0$, $y = y_0$, $\theta = \theta_0$, we can approximate

$$\theta \approx L(x, y) = \theta_0 + \frac{\partial \theta}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial \theta}{\partial y}(y - y_0).$$

Notably, the function L has the same partial derivatives as θ at our point. Since L and θ are changing at the same rates, we can try using L to find the rate of change of θ with respect to t, at a time t_0 at which $x = x_0$ and $y = y_0$:

$$\frac{d\theta}{dt}\bigg|_{t=t_0} = \frac{d}{dt}\left(\underbrace{\frac{\partial \theta}{\partial x}(x_0, y_0)}_{\text{constant}}\underbrace{(x - x_0)}_{x=x(t)} + \underbrace{\frac{\partial \theta}{\partial y}(x_0, y_0)}_{\text{constant}}\underbrace{(y - y_0)}_{y=y(t)}\right)\Bigg|_{t=t_0} = \frac{\partial \theta}{\partial x}(x_0, y_0)\frac{dx}{dt}(t_0) + \frac{\partial \theta}{\partial y}(x_0, y_0)\frac{dy}{dt}(t_0) =$$

$$\left\langle \frac{\partial \theta}{\partial x}(x_0, y_0), \frac{\partial \theta}{\partial y}(x_0, y_0) \right\rangle \cdot \left\langle \frac{dx}{dt}(t_0), \frac{dy}{dt}(t_0) \right\rangle.$$

Note, this works **IF** $\theta = f(x, y)$, $\langle x, y \rangle = \vec{r}(t)$, and the function f is **differentiable** at $(x_0, y_0) = \vec{r}(t_0)$. (Of course $\vec{r}$ must also be differentiable at t_0.) We may call the vector of the partial derivatives of f the *total derivative* of f and write it as f' or as ∇f (the *gradient* of f). Then our formula becomes:

$$(f \circ \vec{r})'(t_0) = f'(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

Definition: If $f : \mathbb{R}^n \to \mathbb{R}$, the *gradient* of f is the vector whose components are its partial derivatives:

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}(x, y, z),\ \frac{\partial f}{\partial y}(x, y, z),\ \frac{\partial f}{\partial z}(x, y, z) \right\rangle.$$

If f is differentiable, we may also call ∇f the (total) derivative of f and write it f'.

Theorem (the chain rule): If $\vec{r}(t)$ is differentiable at t_0, and $f(x, y, z)$ is differentiable at $\vec{r}(t_0)$, then

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = f'(\vec{r}(t)) \cdot \vec{r}\,'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

If you want to picture the chain rule geometrically, here is a way to think about it. Let $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ be the position function of a point moving in the xy-plane. Imagine the xy-plane sitting inside $\mathbb{R}^3$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function whose graph is a surface S, so $f(x, y)$ is the height of the surface at (x, y). Now imagine a point moving on S directly above (or below) the moving point in the xy-plane. The height of that point at time t is given by the composition

$$z = f(\vec{r}(t)) = (f \circ \vec{r})(t).$$

To find how fast this height is changing, we compute

$$\frac{dz}{dt} = (f \circ \vec{r})'(t) = f'(\vec{r}(t)) \cdot \vec{r}\,'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

Example: Suppose a bug is crawling around the surface $z = x^2 + y^2$, so that its shadow is moving in the xy-plane with (x, y)-coordinates at time t given by $\vec{r}(t) = \langle t^2, t^3 \rangle$. When the bug is at the point $(1, 1, 2)$, how fast is its height increasing?

The bug's height is $z = f(x, y) = x^2 + y^2$ when its shadow has position (x, y), and the shadow's position at time t is $(x, y) = \vec{r}(t) = \langle t^2, t^3 \rangle$. When the bug is at $(1, 1, 2)$ we have $\vec{r}(t) = \langle 1, 1 \rangle$ and $t = 1$. By the Chain Rule,

$$\frac{dz}{dt} = \frac{d}{dt}(f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

$$\vec{r}\,'(t) = \langle 2t, 3t^2 \rangle \qquad \vec{r}\,'(1) = \langle 2, 3 \rangle \qquad \nabla f(x, y) = \langle 2x, 2y \rangle \qquad \nabla f(1, 1) = \langle 2, 2 \rangle$$

$$\left.\frac{dz}{dt}\right|_{t=1} = \nabla f(\vec{r}(1)) \cdot \vec{r}\,'(1) = \nabla f(1, 1) \cdot \langle 2, 3 \rangle = \langle 2, 2 \rangle \cdot \langle 2, 3 \rangle = 10.$$

Theorem: (the chain rule): If $\vec{r}(t)$ is differentiable at t_0, and $f(x, y, z)$ is differentiable at $\vec{r}(t_0)$, then

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

Rephrasing this, if w is a function of x, y, z, and x, y, z are all functions of t, then

$$\frac{dw}{dt} = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z \approx \frac{\partial w}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial w}{\partial y}\frac{dy}{dt}\Delta t + \frac{\partial w}{\partial z}\frac{dz}{dt}\Delta t =$$
$$\left(\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}\right)\Delta t = \left(\frac{dw}{dt}\right)\Delta t$$

Here's another way to envision the chain rule physically: Suppose (x, y, z) is the position of a moving object, and $w = f(x, y, z)$ is the temperature at point (x, y, z). To find the rate of change of the temperature of the moving object with respect to time, we have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

We can approximate the change in temperature over a small interval of time Δt by

$$\Delta w \approx \frac{\partial w}{\partial x}\underbrace{\frac{dx}{dt}\Delta t}_{\approx \Delta x} + \frac{\partial w}{\partial y}\underbrace{\frac{dy}{dt}\Delta t}_{\approx \Delta y} + \frac{\partial w}{\partial z}\underbrace{\frac{dz}{dt}\Delta t}_{\approx \Delta z}$$

Example: If $w = x^2y^2$, $x = \sin(t)$, and $y = \cos(t)$, find $\dfrac{dw}{dt}$ at $t = \dfrac{\pi}{3}$.

$$t = \frac{\pi}{3} \qquad x = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad y = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\frac{\partial w}{\partial x} = 2xy^2 = \frac{\sqrt{3}}{4} \qquad \frac{\partial w}{\partial y} = 2x^2y = \frac{3}{4} \qquad \frac{dx}{dt} = \cos(t) = \frac{1}{2} \qquad \frac{dy}{dt} = -\sin(t) = -\frac{\sqrt{3}}{2}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = \left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{4}$$

The chain rule in different settings:

$$w = f(x) = f(g(t))$$

$$t \to x \to w$$

$$\frac{dw}{dt} = \frac{dw}{dx}\,\frac{dx}{dt}$$

$$w = f(x,y,z) = f(\vec{g}(t))$$

$$t \to (x,y,z) \to w$$

$$\frac{dw}{dt} = \underbrace{\frac{\partial w}{\partial x}\,\frac{dx}{dt} + \frac{\partial w}{\partial y}\,\frac{dy}{dt} + \frac{\partial w}{\partial z}\,\frac{dz}{dt}}_{\text{one term for each intermediate variable}}$$

$$w = f(x,y,z) = f(G(s,t))$$

$$(s,t) \to (x,y,z) \to w$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial s}$$

Example: We can identify points on the cone $x^2 + y^2 = z^2$, $z \geq 0$, using two coordinates, r and θ, by setting

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = r \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq r.$$

Define w on the cone by

$$w = xy - xz^2.$$

Find $\dfrac{\partial w}{\partial r}$ at the point $(x, y, z) = (-2, 0, 2)$.

At the point $(x, y) = (-2, 0)$ we have

$$r = 2 \qquad \theta = \pi \qquad x = -2 \qquad y = 0 \qquad z = 2$$

$$\frac{\partial w}{\partial x} = y - z^2 = -4 \qquad \frac{\partial w}{\partial y} = x = -2 \qquad \frac{\partial w}{\partial z} = -2xz = 8$$

$$\frac{\partial x}{\partial r} = \cos(\theta) = -1 \qquad \frac{\partial y}{\partial r} = \sin(\theta) = 0 \qquad \frac{\partial z}{\partial r} = 1$$

We treat θ as a constant and differentiate with respect to r, using the chain rule:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (-4)(-1) + (-2)(0) + (8)(1) = 12$$

At a general point, we have

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y - z^2)(\cos(\theta)) + (x)(\sin(\theta)) + (-2xz)(1) =$$

$$(r\sin(\theta) - r^2)(\cos(\theta)) + (r\cos(\theta))(\sin(\theta)) + (-2r^2\cos(\theta))(1) =$$

$$2r\sin(\theta)\cos(\theta) - 3r^2\cos(\theta).$$

What does this mean? We define w as a function of (r, θ) by looking at the point on the cone $(x, y, z) = (r\cos(\theta), r\sin(\theta), r)$, then computing $w = xy - xz^2$. We want to know, when $(x, y) = (-2, 0)$, the rate of change of w with respect to r.

For example, suppose w denotes the temperature at a given point on the cone. Consider the ubiquitous bug crawling on the cone, with its shadow moving in the xy-plane. The bug's temperature is w. When the bug's shadow is where $(r, \theta) = (2, -\pi)$, and the bug moves so its shadow's new location is where $(r, \theta) = (2 + \Delta r, -\pi)$ (that is, θ remains constant and r changes by Δr), the bug's temperature will have changed by

$$\Delta w \approx \frac{\partial w}{\partial r}\Delta r = 12\Delta r.$$

Recall implictly defined functions: An equation $f(x, y, z) = 0$ defining a surface S can be thought of as implicitly defining z as a function of x and y near a point on S. If we want to find $\frac{\partial z}{\partial x}$ at that point, we can treat y as a constant and z as a function of x, and differentiate the equation wtih respect to x:

$$\frac{\partial}{\partial x}(f(x, y, z)) = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial f}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial f}{\partial z}\underbrace{\frac{\partial z}{\partial x}}_{\text{unknown}} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

Theorem: (the implicit function theorem) If $f(x, y, z) = 0$ implicitly defines z as a function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}.$$

If $f(x, y) = 0$ implicitly defines y as a function of x, then

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

Example: Earlier, we looked at the surface

$$ax^2 + by^2 + cz^2 = d$$

and used implicit differentiation:

$$2ax + 2cz\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{ax}{cz}.$$

Now we can use the implicit function theorem:

$$f(x, y, z) = ax^2 + by^2 + cz^2 - d \qquad f(x, y, z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{2ax}{2cz} = -\frac{ax}{cz}.$$

You do not have to memorize the implicit function theorem, but you may use it if you wish.

Exercise: A surface S has the equation $z = f(x, y)$. At $(x, y) = (1, 2)$ we have

$$z = 45 \qquad \frac{\partial z}{\partial x} = 2 \qquad \frac{\partial z}{\partial y} = 1$$

A bug is crawling on the surface S, and a light shining directly down through S (which is transparent) casts the bug's shadow on the xy-plane; the position of the shadow is $\vec{r}(t)$. At time t_0, the bug's shadow has position $\vec{r}(t_0) = \langle 1, 2 \rangle$ and velocity $\vec{r}\,'(t_0) = \langle 3, -1 \rangle$.

Use the chain rule to find the rate of change of the bug's altitude with respect to time at the time t_0.

Exercise: A radiation source at the origin subjects an object located at point (x, y, z) to radiation of intensity $I = \dfrac{k}{x^2 + y^2 + z^2}$, where k is a constant. At time $t = 0$, an object located at point $(1, 2, 1)$ is moving toward the point $(4, 6, 13)$ at a speed of 2.

Find the object's velocity at time $t = 0$.

At time $t = 0$, at what speed is the radiation intensity experienced by the object changing?

Exercise: We can identify points on the cone $x^2 + y^2 = z^2$, $z \geq 0$, using two coordinates, r and θ, by setting

$$x = r\cos(\theta) \qquad y = r\sin\theta \qquad z = r \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq r.$$

Define w by

$$w = xy - xz^2.$$

Find $\dfrac{\partial w}{\partial \theta}$ at the point $(-2, 0, 2)$.

(This is the same cone, the same function w, and the same point as in the example on page 6.)

Exercise: Suppose that S is a level surface $f(x, y, z) = k$ of a differentiable function f and $\vec{r}(t)$ is a regular parametrization of a path γ lying in S. Since the value of f equals k for all points on S, and all points $\vec{r}(t)$ are on S, we have

$$f(\vec{r}(t)) = k.$$

Start with this equation and differentiate both sides (using the chain rule for the left hand side) to show that

$$\nabla f(\vec{r}(t)) \perp \vec{r}\,'(t).$$

Remember that k is a constant!

Since this is true for any path γ in S, we can conclude that

$$\nabla f \perp S$$

You just proved the following **Theorem:** The gradient of a differentiable function f at a point is normal to the level surface (or level curve) of f containing that point.

Exercise: If $f(x, y, z) = 4x^2 - y^2 + z^2$, then the hyperboloid $4x^2 - y^2 + z^2 = 4$ is a level surface of f, so it should be perpendicular to the gradient of f at every point. The point $(1, 1, 1)$ is on this surface. Verify that the surface is perpendicular to the gradient of f at the point $(1, 1, 1)$ in the following way:

Use the implicit function theorem to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the portion of the surface containing $(1, 1, 1)$, then evaluate those partial derivatives at the point $(1, 1, 1)$. Use these values to find the equation of a tangent plane to the surface at $(1, 1, 1)$.

Verify that the tangent plane is perpendicular to $\nabla f(1, 1, 1)$.

Theorem: (the chain rule): If $\vec{r}(t)$ is differentiable at t_0, and $f(x,y)$ is differentiable at $\vec{r}(t_0)$, then

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

Proof of the Chain Rule:

Let $\vec{r}(t) = (x_0, y_0)$. If f is differentiable at (x_0, y_0), we can express $f(x,y)$ as

$$f(x,y) = \underbrace{a(x-x_0) + b(y-y_0) + f(x_0,y_0)}_{\text{tangent approximation } \mathcal{P}(x,y)} + \underbrace{E(x,y)}_{\text{error } f(x,y)-\mathcal{P}(x,y)}$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{|(x-x_0, y-y_0)|} = 0$$

$$a = \frac{\partial f}{\partial x}(x_0,y_0) \qquad b = \frac{\partial f}{\partial y}(x_0,y_0) \qquad \nabla f(x_0,y_0) = \langle a, b\rangle$$

We have $\vec{r}(t_0) = (x_0, y_0)$ and $\vec{r}$ is differentiable at t_0. We can write $\vec{r}(t) = \langle x(t), y(t)\rangle$ and compute the derivative of $f(\vec{r}(t))$ at $t = t_0$. We use the limit definition of derivative:

$$\frac{d}{dt}\left(f(\vec{r}(t_0))\right) = \lim_{t\to t_0} \frac{f(\vec{r}(t)) - f(\vec{r}(t_0))}{t - t_0}$$

Now we say $\vec{r}(t) = \langle x(t), y(t)\rangle$.

$$\lim_{t\to t_0} \frac{f(\vec{r}(t)) - f(\vec{r}(t_0))}{t - t_0} = \lim_{t\to t_0} \frac{f(x(t), y(t)) - f(x_0, y_0)}{t - t_0}$$

Now we express $f(x,y)$ as above.

$$\lim_{t\to t_0} \frac{f(x(t), y(t)) - f(x_0, y_0)}{t - t_0} =$$

$$\lim_{t\to t_0} \frac{a(x(t)-x_0) + b(y(t)-y_0) + f(x_0,y_0) + E(x(t),y(t)) - f(x_0,y_0)}{t - t_0}$$

We cancel some things out and do some regrouping:

$$\lim_{t\to t_0} \frac{a(x(t)-x_0) + b(y(t)-y_0) + f(x_0,y_0) + E(x(t),y(t)) - f(x_0,y_0)}{t - t_0} =$$

$$\lim_{t\to t_0} \frac{a(x(t)-x(t_0)) + b(y(t)-y(t_0)) + E(x(t),y(t))}{t - t_0} =$$

$$a \lim_{t\to t_0} \frac{(x(t)-x(t_0))}{t-t_0} + b \lim_{t\to t_0} \frac{(y(t)-y(t_0))}{t-t_0} + \lim_{t\to t_0} \frac{E(x(t),y(t))}{t-t_0}$$

In the first two limits we recognize the definition of the derivative.

$$a \lim_{t\to t_0} \frac{(x(t)-x(t_0))}{t-t_0} + b \lim_{t\to t_0} \frac{(y(t)-y(t_0))}{t-t_0} + \lim_{t\to t_0} \frac{E(x(t),y(t))}{t-t_0} =$$

$$ax'(t_0) + by'(t_0) + \lim_{t\to t_0} \frac{E(x(t),y(t))}{t-t_0} =$$

$$\langle a, b\rangle \cdot \langle x'(t_0), y'(t_0)\rangle) + \lim_{t\to t_0} \frac{E(x(t),y(t))}{t-t_0}$$

Now we remember that $\langle a, b\rangle = \nabla f(x_0, y_0) = \nabla f(\vec{r}(t_0))$.

$$\langle a, b\rangle \cdot \langle x'(t_0), y'(t_0)\rangle\rangle + \lim_{t\to t_0} \frac{E(x(t), y(t))}{t - t_0} = \boxed{\nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0)} + \lim_{t\to t_0} \frac{E(x(t), y(t))}{t - t_0}$$

The boxed part is what we want, so we have to show the remaining limit is zero.

We will assume for simplicity that $\vec{r}\,'(t_0) \neq \vec{0}$, so that for t near t_0 we have $\vec{r}(t) \neq \vec{r}(t_0)$, and we can safely divide by $|\vec{r}(t) - \vec{r}(t_0)|$. (This assumption can be eliminated by a small trick.)

$$\lim_{t\to t_0} \left| \frac{E(x(t), y(t))}{t - t_0} \right| = \lim_{t\to t_0} \left| \frac{E(x(t), y(t))}{|\vec{r}(t) - \vec{r}(t_0)|} \right| \left| \frac{|\vec{r}(t) - \vec{r}(t_0)|}{t - t_0} \right| =$$

$$\lim_{t\to t_0} \left| \frac{E(x(t), y(t))}{|\vec{r}(t) - \vec{r}(t_0)|} \right| \lim_{t\to t_0} \left| \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} \right|$$

Now we recognize the definition of derivative in the right-hand term.

$$\lim_{t\to t_0} \left| \frac{E(x(t), y(t))}{|\vec{r}(t) - \vec{r}(t_0)|} \right| \lim_{t\to t_0} \left| \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} \right| = \lim_{t\to t_0} \left| \frac{E(x(t), y(t))}{|\vec{r}(t) - \vec{r}(t_0)|} \right| |\vec{r}\,'(t_0)|$$

We use the fact that as $t \to t_0$ we have $\vec{r}(t) \to \vec{r}(t_0)$, or $(x, y) \to (x_0, y_0)$.

$$\lim_{t\to t_0} \left| \frac{E(x(t), y(t))}{|\vec{r}(t) - \vec{r}(t_0)|} \right| |\vec{r}\,'(t_0)| = \lim_{(x,y)\to(x_0,y_0)} \left| \frac{E(x, y)}{|(x, y) - (x_0, y_0)|} \right| |\vec{r}\,'(t_0)|$$

Now we use the definition of tangent.

$$\left(\lim_{(x,y)\to(x_0,y_0)} \left| \frac{E(x, y)}{|(x, y) - (x_0, y_0)|} \right| \right) |\vec{r}\,'(t_0)| = (0) \left(|\vec{r}\,'(t_0)|\right) = 0.$$

This is what we wanted. We have shown

$$\boxed{\frac{d}{dt} \left(f(\vec{r}(t_0))\right) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0)}$$