Math 8
Winter 2020
Section 1
February 27, 2020

First, some important points from the last class:

Definition: If $f : \mathbb{R}^n \to \mathbb{R}$, the *gradient* of f is the vector whose components are its partial derivatives:

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}(x,y,z),\ \frac{\partial f}{\partial y}(x,y,z),\ \frac{\partial f}{\partial z}(x,y,z) \right\rangle.$$

If f is differentiable, we may also call ∇f the total derivative of f and write it as f'.

Theorem (the chain rule): If $\vec{r}(t)$ is differentiable at t_0, and $f(x,y,z)$ is differentiable at $\vec{r}(t_0)$, then

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = f'(\vec{r}(t)) \cdot \vec{r}\,'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

$$\frac{dw}{dt} = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z \approx \frac{\partial w}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial w}{\partial y}\frac{dy}{dt}\Delta t + \frac{\partial w}{\partial z}\frac{dz}{dt}\Delta t$$

The chain rule in different settings:

$$t \to x \to w$$
$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

$$t \to (x,y,z) \to w$$
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$(s,t) \to (x,y,z) \to w$$
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

Theorem (the implicit function theorem): An equation $f(x,y,z) = 0$ defining a surface S can be thought of as implicitly defining z as a function of x and y near a point on S. Then we have

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}.$$

Example: Let $g(x,y) = 2 - x^2$.

Explain why

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{g((1,0)+(\Delta x,\Delta y)) - g((1,0))}{(\Delta x,\Delta y)}$$

could not possibly give us the slope of the graph of g when $x = 1$ and $y = 0$.

Answer: You can't divide by $(\Delta x, \Delta y)$.

Might it be that

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{g((1,0)+(\Delta x,\Delta y)) - g((1,0))}{|(\Delta x,\Delta y)|}$$

gives us the slope of the graph of g when $x = 1$ and $y = 0$?

Answer: $$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{g((1,0)+(\Delta x,\Delta y)) - g((1,0))}{|(\Delta x,\Delta y)|}$$

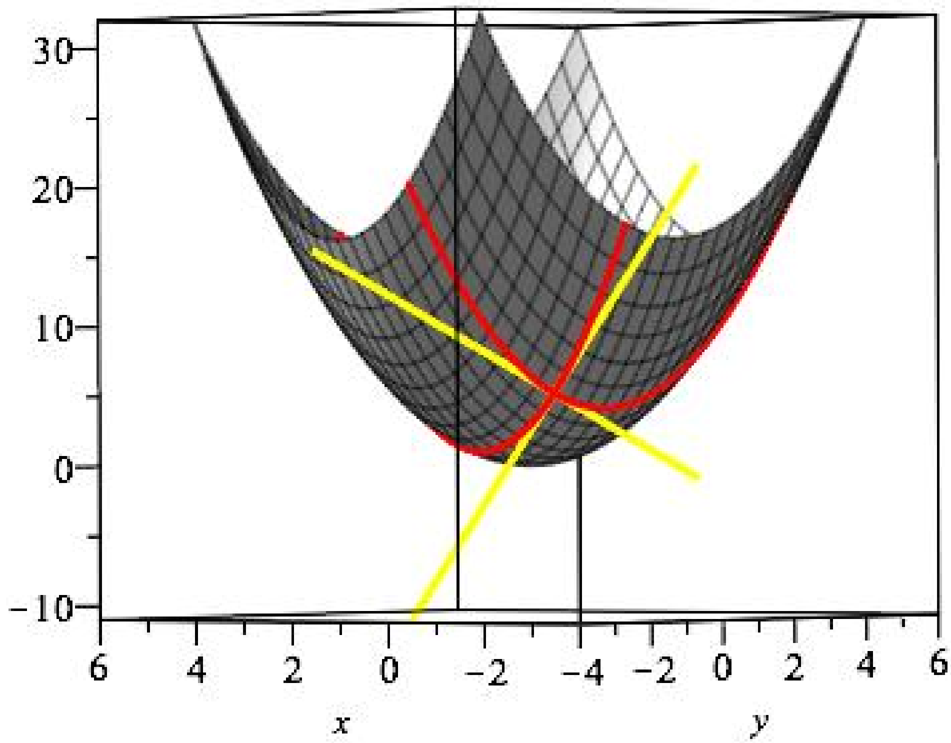
$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{-(\Delta x)^2 - 2\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} =$$

$$\lim_{(u,v)\to(0,0)} \frac{-u^2 - 2u}{\sqrt{u^2+v^2}}$$

which does not exist.

(In polar coordinates we have $\dfrac{-r^2\cos^2\theta - 2r\cos\theta}{r} = -r\cos^2\theta - 2\cos\theta$.) So this does not give us the slope.

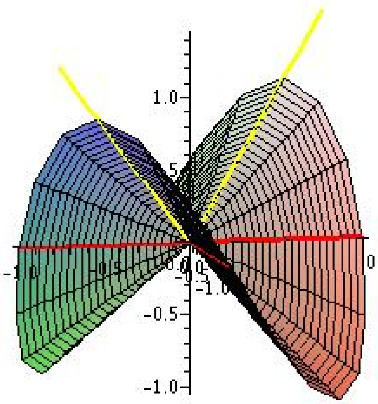
Here is the problem. This picture shows the graph of the function $f(x, y) = x^2 + y^2$.

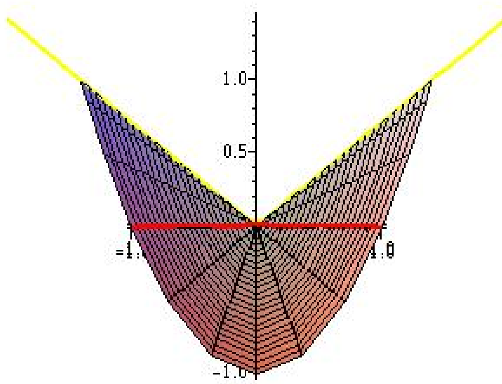


We can try to find the slope of this graph at some point by looking at a path on the graph going through that point and finding the slope of the path; that is, by finding the slope of the tangent line to the path, as (vertical) rise over (horizontal) run. The two red curves are paths going through the same point (slices by different vertical planes), and the yellow lines are the tangent lines to those curves at that point. One line is steeper than the other. The graph has different slopes in different directions.

Instead of looking for a slope, we will look for a tangent plane $z = ax + by + d$. Since the numbers a and b completely determine the slope of the plane in any direction, we will say the derivative of our function is this pair of numbers, $\langle a, b\rangle$.

In the picture above, we have found two tangent lines, so it looks like we have found a tangent plane, namely, the plane containing those two lines. In that picture, we have. However, it's not always that simple.





The surface pictured above (from two different points of view), the graph of the function $f(x, y) = \dfrac{2xy}{\sqrt{x^2 + y^2}}$, contains the x- and y-axes (red lines in the picture), so we might think the xy-plane is tangent to the surface at $(0, 0, 0)$. However, a vertical slice of the surface by the plane $y = x$ (drawn in yellow) has a sharp point at $(0, 0, 0)$. Therefore, this surface does not have a tangent plane at $(0, 0, 0)$.

We will need to consider what "tangent plane" means.

Definition: If the graphs of $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}$ are tangent at the point (x_0, y_0, z_0), and

$$\mathcal{P}(x,y) = ax + by + d = \langle a, b\rangle \cdot \langle x, y\rangle + d$$

(in other words, the graph of $\mathcal{P}$ is a tangent plane to the graph of f), then we say f is differentiable at (x_0, y_0) and

$$\boxed{f'(x_0, y_0) = \langle a, b\rangle \,.}$$

Note: We still need to define what "tangent" means.

Note: This is just like the case for $f : \mathbb{R} \to \mathbb{R}$. If the graph of the function

$$\ell(x) = ax + d$$

is the tangent line to the graph of f at the point (x_0, y_0), then the derivative of f at that point is the slope of that line:

$$\boxed{f'(x_0) = a.}$$

This is also just like the case $\vec{f} : \mathbb{R} \to \mathbb{R}^n$: The tangent approximation to $\vec{f}$ at $t = t_0$ is

$$\vec{r}(t) = (t - t_0)\vec{v} + \vec{r}_0 = t\vec{v} + \underbrace{(\vec{r}_0 - t_0\vec{v})}_{\vec{d}}$$

where

$$\boxed{\vec{f}\,'(t_0) = \vec{v}.}$$

General idea: Suppose f is a function, and $\mathcal{T}$ is a function of the form

$$\mathcal{T}(x) = Ax + D$$

where A and D are constants of the appropriate type (scalars or vectors), and multiplication can mean ordinary multiplication, scalar multiplication, or dot product, as appropriate. (The input x may also be a scalar or a vector.) If the graphs of f and $\mathcal{T}$ are tangent where $x = x_0$, then

$$\boxed{f'(x_0) = A.}$$

Other functions: The three types of multiplication listed here can be thought of as special cases of one kind of multiplication, called matrix multiplication. With the help of matrix multiplication, you can define derivatives for $f : \mathbb{R}^m \to \mathbb{R}^n$ in the same way.

We won't talk about matrix multiplication in Math 8. Instead, we will take derivatives of functions $F : \mathbb{R}^m \to \mathbb{R}^n$ coordinatewise: if $F(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$, then $F'(x,y) = \langle F_1'(x,y), F_2'(x,y)\rangle$. This is not quite right (this is a vector of vectors, which we really should arrange into a matrix), but thinking coordinatewise will let us apply derivatives to find tangent approximations and other things.

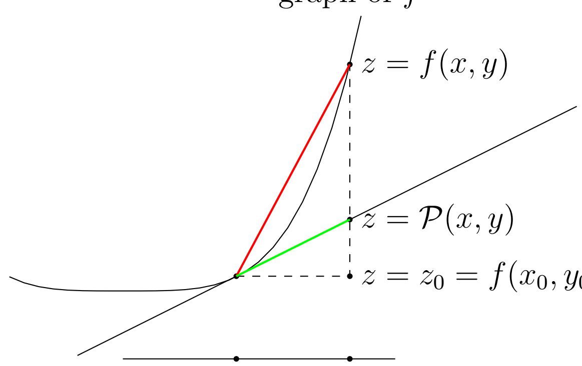
Now we need to say what "tangent" means.

Definition: We say the graphs of f and $\mathcal{P}$ are tangent at (x_0, y_0, z_0) provided two things hold:

1. The point (x_0, y_0, z_0) is on both graphs.

2. As $(x, y) \to (x_0, y_0)$ the slopes of secant lines on the two graphs, with one end point at (x_0, y_0) and the other at (x, y), approach each other:

$$\lim_{(x,y)\to(x_0,y_0)} \big[(\text{slope of secant line to graph of } f) - (\text{slope of secant line to graph of } \mathcal{P})\big] = 0.$$



The picture shows a vertical slice of the graphs of f and $\mathcal{P}$, including the point (x_0, y_0, z_0) at which they are tangent. The horizontal line at the bottom is a slice of the xy-plane.

Since (x_0, y_0, z_0) is on both graphs, we have $z_0 = f(x_0, y_0) = \mathcal{P}(x_0, y_0)$.

The slopes of the two secant lines are (vertical) rise over (horizontal) run. In both cases, the horizontal run is the distance between (x, y) and (x_0, y_0), which is $|(x, y) - (x_0, y_0)|$. The vertical rise is the difference in z-coordinates, which is $f(x, y) - f(x_0, y_0) = f(x, y) - z_0$ for the (red) secant line to the graph of f, and $\mathcal{P}(x, y) - \mathcal{P}(x_0, y_0) = \mathcal{P}(x, y) - z_0$ for the (green) secant line to the graph of $\mathcal{P}$.

This lets us express part (2) as:

$$\lim_{(x,y)\to(x_0,y_0)} (\text{slope of secant line to graph of } f) - (\text{slope of secant line to graph of } \mathcal{P}) = 0.$$

$$\lim_{(x,y)\to(x_0,y_0)} \Bigg[\underbrace{\left(\frac{f(x, y) - f(x_0, y_0)}{|(x, y) - (x_0, y_0)|} \right)}_{\text{slope of secant to graph of } f} - \underbrace{\left(\frac{\mathcal{P}(x, y) - \mathcal{P}(x_0, y_0)}{|(x, y) - (x_0, y_0)|} \right)}_{\text{slope of secant to graph of } \mathcal{P}} \Bigg] = 0.$$

$$\lim_{(x,y)\to(x_0,y_0)} \left[\left(\frac{f(x, y) - z_0}{|(x, y) - (x_0, y_0)|} \right) - \left(\frac{\mathcal{P}(x, y) - z_0}{|(x, y) - (x_0, y_0)|} \right) \right] = 0;$$

$$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x, y) - \mathcal{P}(x, y)}{|(x, y) - (x_0, y_0)|} \right] = 0.$$

So, putting this together:

Definition: A function $f : \mathbb{R}^2$ to $\mathbb{R}$ is differentiable at (x_0, y_0) if there is a function $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\mathcal{P}(x, y) = ax + by + d = \langle a, b \rangle \cdot \langle x, y \rangle + d$$

whose graph is tangent to the graph of f where $(x, y) = (x_0, y_0)$. In this case we say

$$f'(x_0, y_0) = \langle a, b \rangle .$$

Tangent means

$$f(x_0, y_0) = \mathcal{P}(x_0, y_0) \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - \mathcal{P}(x, y)}{|(x, y) - (x_0, y_0)|} = 0.$$

Note: We can write

$$\mathcal{P}(x, y) = f(x_0, y_0) + f'(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle .$$

This is the tangent approximation to f at that point.

Definition: A function $f : \mathbb{R}^3$ to $\mathbb{R}$ is differentiable at (x_0, y_0, z_0) if there is a function $\mathcal{P} : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\mathcal{P}(x, y, z) = ax + by + cz + d = \langle a, b, c \rangle \cdot \langle x, y, z \rangle + d$$

whose graph is tangent to the graph of f where $(x, y, z) = (x_0, y_0, z_0)$. In this case we say

$$f'(x_0, y_0, z_0) = \langle a, b, c \rangle .$$

Tangent means

$$f(x_0, y_0, z_0) = \mathcal{P}(x_0, y_0, z_0) \quad \text{and} \quad \lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{f(x, y, z) - \mathcal{P}(x, y, z)}{|(x, y, z) - (x_0, y_0, z_0)|} = 0.$$

Note: We can write

$$\mathcal{P}(x, y, z) = f(x_0, y_0, z_0) + f'(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle .$$

This is the tangent approximation to f at that point.

Example: Suppose $f(x,y) = x^2 + y$. Use the limit definition to show that f is differentiable at $(1,2)$.

We need to find the equation of a potential tangent plane, and then use the limit definition to show the graphs are actually tangent.

The potential tangent plane is the graph of the linearization of f:

$$\mathcal{P}(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-1).$$

We can compute $f_x(x,y) = 2x$ and $f_y(x,y) = 1$, so $f_x(1,2) = 2$ and $f_y(1,2) = 1$. This gives

$$\mathcal{P}(x,y) = 3 + 2(x-1) + 1(y-2) = 2x + y - 1.$$

Now we have that $(1,2,3)$ is on both graphs. To show they are tangent at that point we must check the limit:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y)-(x_0,y_0)|} = \lim_{(x,y)\to(1,2)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y)-(1,2)|} = \lim_{(x,y)\to(1,2)} \frac{x^2 + y - (2x + y - 1)}{\sqrt{(x-1)^2 + (y-2)^2}} =$$

$$\lim_{(x,y)\to(1,2)} \frac{x^2 - 2x + 1}{\sqrt{(x-1)^2 + (y-2)^2}} = \lim_{(x,y)\to(1,2)} \frac{(x-1)^2}{\sqrt{(x-1)^2 + (y-2)^2}} =$$

$$\lim_{(x,y)\to(1,2)} |x-1| \frac{|x-1|}{\sqrt{(x-1)^2 + (y-2)^2}} = \lim_{(x,y)\to(1,2)} |x-1| \frac{\sqrt{(x-1)^2}}{\sqrt{(x-1)^2 + (y-2)^2}} =$$

$$\lim_{(x,y)\to(1,2)} \underbrace{|x-1|}_{\text{approaches } 0} \underbrace{\sqrt{\frac{(x-1)^2}{(x-1)^2 + (y-2)^2}}}_{\text{between 0 and 1}} = 0$$

Alternative approach: Set $x = 1 + \Delta x$ and $y = 2 + \Delta y$, so as $(x,y) \to (1,2)$ we have $(\Delta x, \Delta y) \to (0,0)$. This converts the limit to

$$\lim_{(x,y)\to(1,2)} \frac{(x-1)^2}{\sqrt{(x-1)^2 + (y-2)^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{(\Delta x)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}},$$

which we can analyze using polar coordinates, writing $\Delta x = r\cos\theta$ and $\Delta y = r\sin\theta$.

Exercise: Let $f(x, y) = x^2 + 3y^2$ and show that f is differentiable at $(2, 1)$.

Exercise: Let $f(x, y) = \sin(x^2 + y^2)$ and show that $f'(0, 0) = \langle 0, 0\rangle$ using the limit definition.

To do this, you should find the equation of the candidate tangent plane, and use the limit definition to show that plane actually is tangent to the graph of f at $(x, y) = (0, 0)$.

Exercise: Use geometrical reasoning to figure out the equation of the plane tangent to the graph of the function

$$f(x, y) = \sqrt{1 - x^2 - y^2}$$

at the point $(0, 0, 1)$. (Hint: This is the top half of the unit sphere.) Then use the definition to prove this is really a tangent plane.

Small challenge: Consider the function $f(x, y) = e^{\sqrt{x^2+y^2}}$. Note that $f(0, 0) = 1$.

Argue by the symmetry of the function that the only possible tangent plane at $(0, 0)$ is the horizontal plane $z = 1$.

Use the limit definition of tangent plane to show this plane is *not* tangent to the graph of f at the point $(0, 0, 1)$.

Draw the intersection of the graph of f with the plane $y = 0$. Does this confirm the idea that f is not differentiable at $(0, 0)$?

Larger challenge: Part of your work above was to prove a certain limit does not equal 0. Try to prove this using the formal ε-δ definition of limit.