Math 8
Winter 2020
Section 1
March 2, 2020

First, some important points from the last class:

Definition: If $f : \mathbb{R}^n \to \mathbb{R}$ and $\vec{u} = \langle u_1, \ldots, u_n \rangle$ is a unit vector in $\mathbb{R}^n$, then the *directional derivative* of f at $(x_1, \ldots, x_n)$ in the direction $\vec{u}$ is

$$D_{\vec{u}} f(x_1, \ldots, x_n) = \frac{\partial f}{\partial \vec{u}}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f((x_1, \ldots, x_n) + h(u_1, \ldots, u_n)) - f(x_1, \ldots, x_n)}{h}.$$

This is the rate of change of f with respect to distance, when the argument (input) of f is moving in the direction $\vec{u}$.

If $f : \mathbb{R}^2 \to \mathbb{R}$, then $D_{\vec{u}}(x, y)$ is the slope of the slice of the graph of f in the vertical plane containing the line in the xy-plane through the point (x, y) in the direction of the vector $\vec{u}$.

If f denotes temperature in degrees, and we measure distances in meters, then the units of $D_{\vec{u}}(f)$ are degrees per meter.

Theorem: If f is differentiable at $(x_1, \ldots, x_n)$, then

$$D_{\vec{u}}(x_1, \ldots, x_n) = \nabla f(x_1, \ldots, x_n) \cdot \vec{u}.$$

Warning: The vector $\vec{u}$ must be a unit vector.

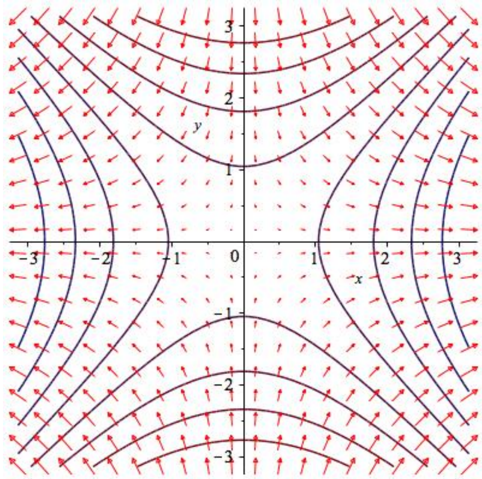
Theorem: If f is differentiable at $(x_1, \ldots, x_n)$ then:

The maximum value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is $|\nabla f(x_1, \ldots, x_n)|$ and it occurs when $\vec{u}$ points in the direction of $\nabla f(x_1, \ldots, x_n)$.

The minimum value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is $-|\nabla f(x_1, \ldots, x_n)|$ and it occurs when $\vec{u}$ points in the opposite direction to $\nabla f(x_1, \ldots, x_n)$.

The value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is 0 when $\vec{u}$ is perpendicular to $\nabla f(x_1, \ldots, x_n)$.

The vector $\nabla f(x_1, \ldots, x_n)$ is normal to the level set (level curve or level surface) of f containing the point $(x_1, \ldots, x_n)$.



$f(x, y) = x^2 - y^2$

Here is a contour plot, and a picture of the *gradient field*, of the function

$$f(x, y) = x^2 - y^2.$$

The gradient field is an example of a vector field, a function that assigns to every point a vector. In this case, ∇f assigns to every point (x, y) the vector $\nabla f(x, y)$.

Remember that level curves of f are in the domain of f. In this case, for $f : \mathbb{R}^2 \to \mathbb{R}$, the graph of f is a surface in $\mathbb{R}^3$, but the level curves of f are curves in the domain $\mathbb{R}^2$.

Gradient vectors of f are also in the domain of f.

Preliminary homework: Find all points where the graph of the function

$$f(x,y) = 3x - x^3 - 2y^2 + y^4$$

has a horizontal tangent plane.

This is a polynomial, so it is differentiable everywhere, and it will have a horizontal tangent plane exactly when both partial derivatives equal zero.

$$\frac{\partial f}{\partial x}(x,y) = 3 - 3x^2 = 3(1-x^2) \qquad \frac{\partial f}{\partial y}(x,y) = -4y + 4y^3 = 4y(y^2-1)$$

$$\frac{\partial f}{\partial x}(x,y) = 0 \text{ when } x = 1 \text{ or } x = -1 \qquad \frac{\partial f}{\partial y}(x,y) = 0 \text{ when } y = 0 \text{ or } y = 1 \text{ or } y = -1$$

The tangent plane is horizontal at the points

$$(1,0) \quad (1,1) \quad (1,-1) \quad (-1,0) \quad (-1,1) \quad (-1,-1).$$

Definition: The point (a, b) is a *critical point* of $f(x, y)$ if either $\nabla f(a, b) = \langle 0, 0\rangle$ or $\nabla f(a, b)$ is undefined.

The point (a, b) is a *local maximum point* of $f(x, y)$ if there is any neighborhood of (a, b) throughout which $f(x, y) \leq f(a, b)$. (A neighborhood of (x, y) is a disc centered at (x, y).)

The point (a, b) is a *local minimum point* of $f(x, y)$ if there is any neighborhood of (a, b) throughout which $f(x, y) \geq f(a, b)$.

The point (a, b) is a *saddle point* of $f(x, y)$ if $\nabla f(a, b) = \langle 0, 0\rangle$ and (a, b) is neither a local maximum point nor a local minimum point.

Theorem: Local maximum and minimum points are always critical points.

Note: This applies to functions of more than two variables as well.

Question: For the function $f(x, y) = 3x - x^3 - 2y^2 + y^4$, can we tell which of the six critical points are local minimum points, local maximum points, and saddle points?

Note that

$$f(x, y) = 3x - x^3 - 2y^2 + y^4 = g(x) + h(y) \text{ where } g(x) = 3x - x^3 \text{ and } h(y) = y^4 - 2y^2.$$

We can analyze g and h:

$$g'(x) = 3 - 3x^2 \qquad g''(x) = -6x \qquad g''(-1) = 6 > 0 \qquad g''(1) = -6 < 0,$$

so by the second derivative test, $x = -1$ is a local minimum point and $x = 1$ is a local maximum point for $g(x)$.

$$h'(y) = 4y^3 - 4y \qquad h''(y) = 12y^2 - 4 \qquad h''(-1) = h''(1) = 8 > 0 \qquad h''(0) = -4.$$

By the second derivative test, $y = -1$ and $y = 1$ are local minimum points for $h(y)$, and $y = 0$ is a local maximum point.

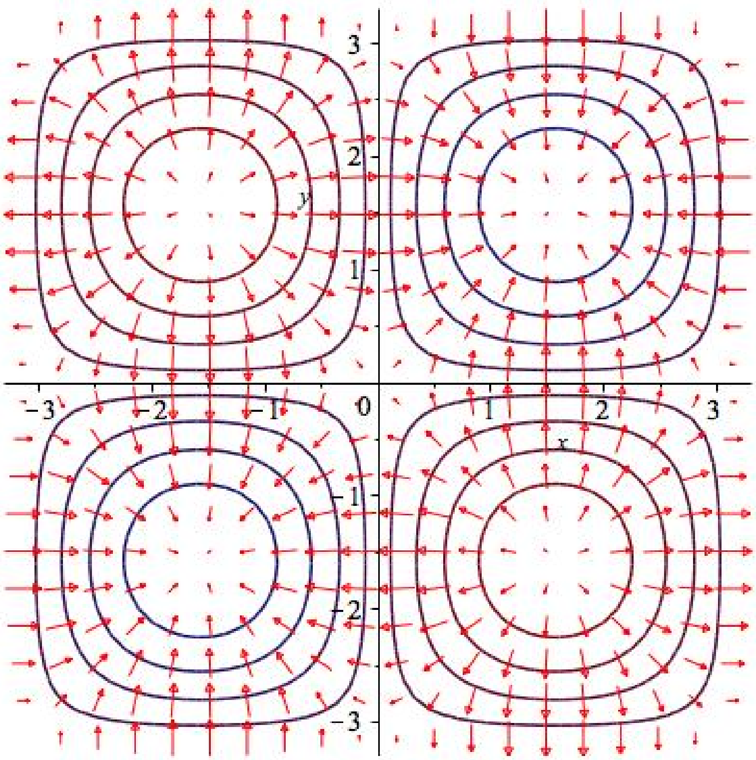
The point $(1, 0)$, where both $g(x)$ and $h(y)$ reach local maxima, is a local maximum point for f.

The points $(-1, -1)$ and $(-1, 1)$, where both $g(x)$ and $h(y)$ reach local minima, are local minimum points for f.

The points $(1, -1)$, $(1, 1))$, and $(-1, 0)$, where one of the functions reaches a local maximum and the other reaches a local minimum, are saddle points for f.

Usually it is not this easy, because a function $f(x, y)$ cannot usually be written in the form $g(x) + h(y)$.

Example: Here are some level curves, and the gradient field, of $f(x, y) = \sin(x)\sin(y)$. Where do we see critical points? Are they local maxima, local minima, or saddle points?



Definition: If (a, b) is a critical point of $f(x, y)$, and all the second partial derivatives of f are defined and continuous near (a, b), we define the discriminant of f at (a, b) to be

$$D(a,b) = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2}(a,b) & \dfrac{\partial^2 f}{\partial x \partial y}(a,b) \\ \\ \dfrac{\partial^2 f}{\partial y \partial x}(a,b) & \dfrac{\partial^2 f}{\partial y^2}(a,b) \end{vmatrix}$$

Theorem (the second derivative test): If (a, b) is a critical point of $f(x, y)$, and all the second partial derivatives of f are defined and continuous near (a, b), then

$$D(a,b) > 0 \quad \Longrightarrow \quad (a,b) \text{ is a local minimum or maximum point;}$$

$$D(a,b) < 0 \quad \Longrightarrow \quad (a,b) \text{ is a saddle point;}$$

$$D(a,b) = 0 \quad \Longrightarrow \quad \text{the second derivative test fails to give any information about (a,b).}$$

$$D(a,b) > 0 \;\;\&\;\; \frac{\partial^2 f}{\partial x^2}(a,b) < 0 \Longrightarrow (a,b) \text{ is a local maximum point;}$$

$$D(a,b) > 0 \;\;\&\;\; \frac{\partial^2 f}{\partial x^2}(a,b) > 0 \Longrightarrow (a,b) \text{ is a local minimum point.}$$

Note: This second derivative test is for functions $f : \mathbb{R}^2 \to \mathbb{R}$. We are not learning a second derivative test for functions $f : \mathbb{R}^3 \to \mathbb{R}$.

Example: Apply the second derivative test to some of the critical points of the function from the preliminary homework:

$$f(x,y) = 3x - x^3 - 2y^2 + y^4$$

$$\frac{\partial f}{\partial x}(x,y) = 3 - 3x^2 = 3(1-x^2) \qquad \frac{\partial f}{\partial y}(x,y) = -4y + 4y^3 = 4y(y^2-1)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -6x \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = 12y^2 - 4 \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = 0$$

$$D(x,y) = \begin{vmatrix} -6x & 0 \\ 0 & 12y^2 - 4 \end{vmatrix} = -6x(12y^2 - 4)$$

$$D(1,0) = -6(-4) = 24 > 0 \ \ \& \ \ \frac{\partial^2 f}{\partial x^2}(1,0) = -6 < 0 \quad \text{so } (1,0) \text{ is a local maximum point.}$$

$$D(1,1) = -6(8) = -48 < 0 \quad \text{so } (1,1) \text{ is a saddle point.}$$

Example: Find the critical points of

$$f(x,y) = 2x^3 - x^2 y + y$$

and use the second derivative test to classify them as local maximum points, local minimum points, or saddle points.

Example: The only critical point of $f(x, y) = x^2 + y^2$ is $(0, 0)$, which is a local minimum point.

What are the largest and smallest values of $f(x, y)$ on the square region D defined by $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, and where are they located?

Since $f(x, y)$ is the square of the distance from the origin to (x, y), we can analyze this one without too much trouble.

The smallest value is $f(0, 0) = 0$.

The largest value is $f(1, 1) = f(1, -1) = f(-1, 1) = f(-1, -1) = 2$.

The smallest value occurs inside D, at a critical point of f. The largest value occurs at the edge of D. This illustrates the general case.

Definition: A region D is *bounded* if there is some number b such that every point in D has a distance from the origin of at most b.

D is *open* if every point that belongs to D has a neighborhood that is included in D.

D is *closed* if every edge point of D belongs to D. (In three dimensions, every point on the surface of D belongs to D.)

Example: The region $x^2 + y^2 < 1$ is open and bounded.

The region $x^2 + y^2 \leq 1$ is closed and bounded.

The region $1 < x^2 + y^2 \leq 4$ is bounded, and neither closed nor open.

The region $x^2 + y^2 \geq 1$ is closed and unbounded.

Definition: The number c is an *absolute maximum* value for $f(x, y)$ on D if there is some point (a, b) in D such that $f(a, b) = c$, and for all points (x, y) in D we have $f(x, y) \leq c$. The absolute maximum value c is attained at (a, b).

The number c is an *absolute minimum* value for $f(x, y)$ on D if there is some point (a, b) in D such that $f(a, b) = c$, and for all points (x, y) in D we have $f(x, y) \geq c$. The absolute minimum value c is attained at (a, b).

Theorem: A continuous function $f(x, y)$ defined on a closed bounded region D has an absolute maximum value and an absolute minimum value on D. The points at which those extreme values are attained are either critical points of f or edge points of D.

Note: This applies to functions of more than two variables as well.

Example: Find the largest and smallest values of $f(x,y) = x^2 - y^2$ on the region $x^2 + y^2 \leq 1$.

There is one critical point of f, the origin $(0,0)$, and

$$f(0,0) = 0.$$

This is a possible candidate for the largest or smallest value.

Now we have to check the edge points.

Method 1: Parametrize the edge, by
$(x,y) = (\cos(t), \sin(t))$, $0 \leq t \leq 2\pi$, and find the largest and smallest values of $f(\cos(t), \sin(t))$.

$$g(t) = f(\cos(t), \sin(t)) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t)$$
$$g'(t) = -4\sin(t)\cos(t)$$

Check critical points of g and end points of the interval. End points: $t = 0$ ($(x,y) = (1,0)$), $t = 2\pi$ ($(x,y) = (1,0)$). Critical points other than end points: When $\sin(t) = 0$, $t = \pi$, $(x,y) = (-1,0)$. When $\cos(t) = 0$, $t = \frac{\pi}{2}$ ($(x,y) = (0,1)$), $t = \frac{3\pi}{2}$ ($(x,y) = (0,-1)$). This gives these possible candidates for maximum or minimum value of f:

$$f(1,0) = 1 \qquad f(-1,0) = 1 \qquad f(0,1) = -1 \qquad f(0,-1) = -1.$$

Compare these to the value at our critical point, $f(0,0) = 0$.

The maximum value is $f(1,0) = f(-1,0) = 1$, and the minimum value is $f(0,1) = f(0,-1) = -1$.

Method 2: Write y in terms of x on the top half of the circle:

$$y = \sqrt{1-x^2} \qquad -1 \leq x \leq 1 \quad f(x,y) = f(x, \sqrt{1-x^2}) = x^2 - (1-x^2) = 2x^2 - 1 = h(x).$$

Now find the largest and smallest values of $h(x)$ by checking critical points and end points.

Critical point: $h'(x) = 4x = 0$ when $x = 0$ ($(x,y) = (0,1)$).

End points: $x = -1$ ($(x,y) = (-1,0)$), $x = 1$ ($(x,y) = (1,0)$).

This gives $(0,1)$, $(-1,0)$,and $(1,0)$ as candidate edge points at which f could reach its maximum or minimum value.

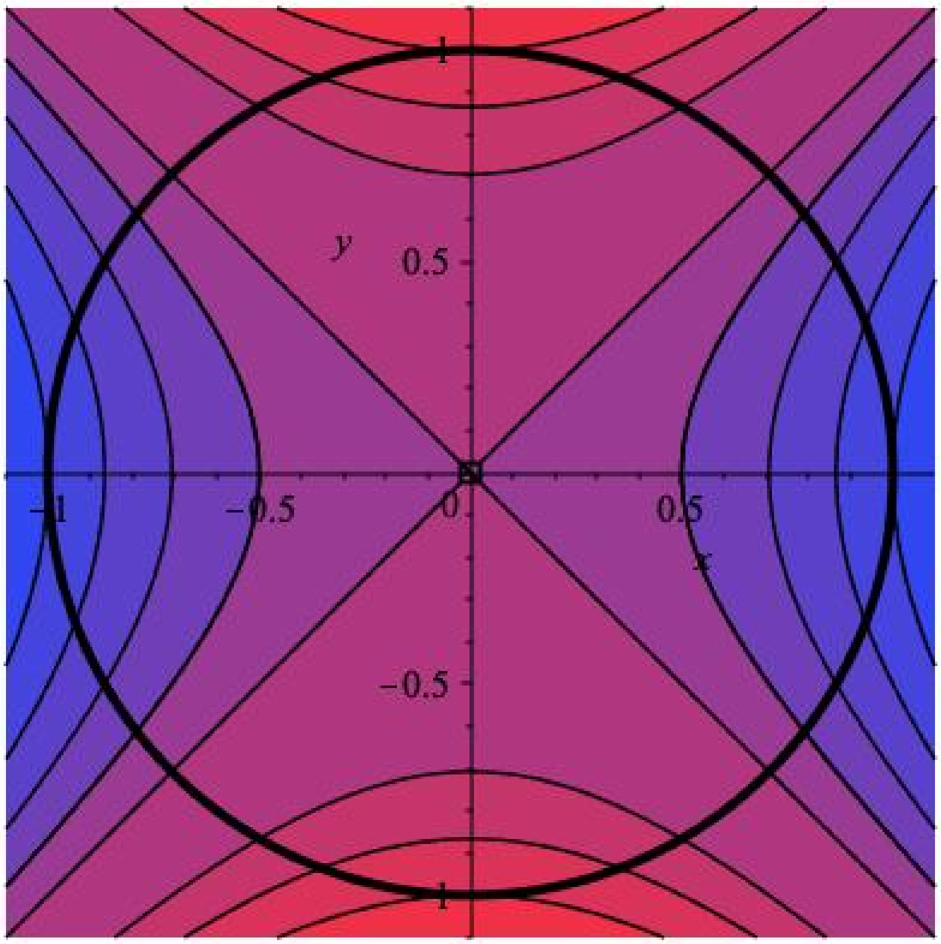
Doing the same on the bottom half of the circle gives $(0,-1)$, $(-1,0)$,and $(1,0)$.

Now we have the same five points to check as before:

$$f(1,0) = 1 \qquad f(-1,0) = 1 \qquad f(0,1) = -1$$
$$f(0,-1) = -1 \qquad f(0,0) = 0.$$

Method 3: We'll see another way to check edge points next class.

Here is a contour plot of the function $f(x, y) = x^2 - y^2$ from the previous example, with the edge of the region $x^2 + y^2 \leq 1$ drawn in thick black. Red regions represent lower values of f and blue regions represent higher values.



Exercise: Find all critical points of the function

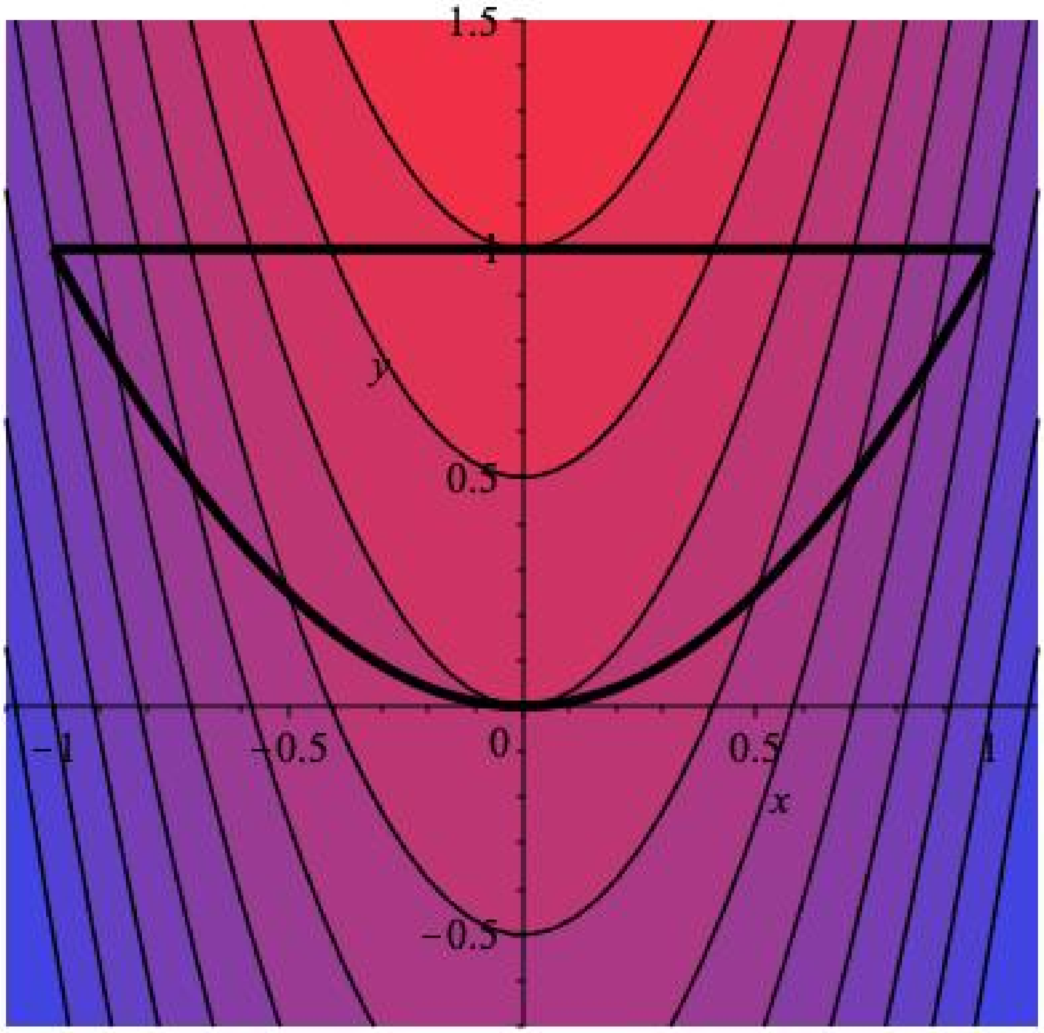
$$f(x, y) = x^3 + 3xy + y^2 + 2y$$

and classify each of them as a local maximum point, local minimum point, or saddle point.

Exercise: Use the second derivative test to classify the remaining critical points $(1, -1)$, $(-1, 0)$, $(-1, 1)$, $(-1, -1)$ of the function $f(x, y) = 3x - x^3 - 2y^2 + y^4$ from the preliminary homework.

Exercise: Find the largest and smallest values of the function $f(x, y) = 3x^2 - y$ on the region $x^2 \leq y \leq 1$, and the points at which these values are obtained. (Hint: Draw this region first.)

On the next page is a contour plot of the function $f(x, y) = 3x^2 - y$ from this exercise. Red regions represent lower values of f and blue regions represent higher values. The boundary of the region $x^2 \leq y \leq 1$ is drawn in thick black lines. You should confirm that your answer to this problem agrees with the picture.


1.5
1
y
0.5
−0.5
−1
−0.5
0
0.5
1
x