Math 8
Winter 2020
Section 1
March 4, 2020

First, some important points from the last class:

Definition: The point (a, b) is a *critical point* of $f(x, y)$ if either $\nabla f(a, b) = \langle 0, 0\rangle$ or $\nabla f(a, b)$ is undefined.

The point (a, b) is a *local maximum point* of $f(x, y)$ if there is any neighborhood of (a, b) throughout which $f(x, y) \le f(a, b)$.

The point (a, b) is a *local minimum point* of $f(x, y)$ if there is any neighborhood of (a, b) throughout which $f(x, y) \ge f(a, b)$.

The point (a, b) is a *saddle point* of $f(x, y)$ if $\nabla f(a, b) = \langle 0, 0\rangle$ and (a, b) is neither a local maximum point nor a local minimum point.

Theorem: Local maximum and minimum points are always critical points.

Definition: If (a, b) is a critical point of $f(x, y)$, and all the second partial derivatives of f are defined and continuous near (a, b), we define the discriminant of f at (a, b) to be

$$D(a,b) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{vmatrix}$$

Theorem (the second derivative test): If (a, b) is a critical point of $f(x, y)$, and all the second partial derivatives of f are defined and continuous near (a, b), then

$$D(a,b) > 0 \ \ \& \ \ \frac{\partial^2 f}{\partial x^2}(a,b) < 0 \implies (a,b) \text{ is a local maximum point;}$$

$$D(a,b) > 0 \ \ \& \ \ \frac{\partial^2 f}{\partial x^2}(a,b) > 0 \implies (a,b) \text{ is a local minimum point;}$$

$$D(a,b) < 0 \implies (a,b) \text{ is a saddle point;}$$

$$D(a,b) = 0 \implies \text{the second derivative test fails to give any information about (a,b).}$$

Definition: A region D is *bounded* if there is some number b such that every point in D has a distance from the origin of at most b.

D is *open* if every point that belongs to D has a neighborhood that is included in D.

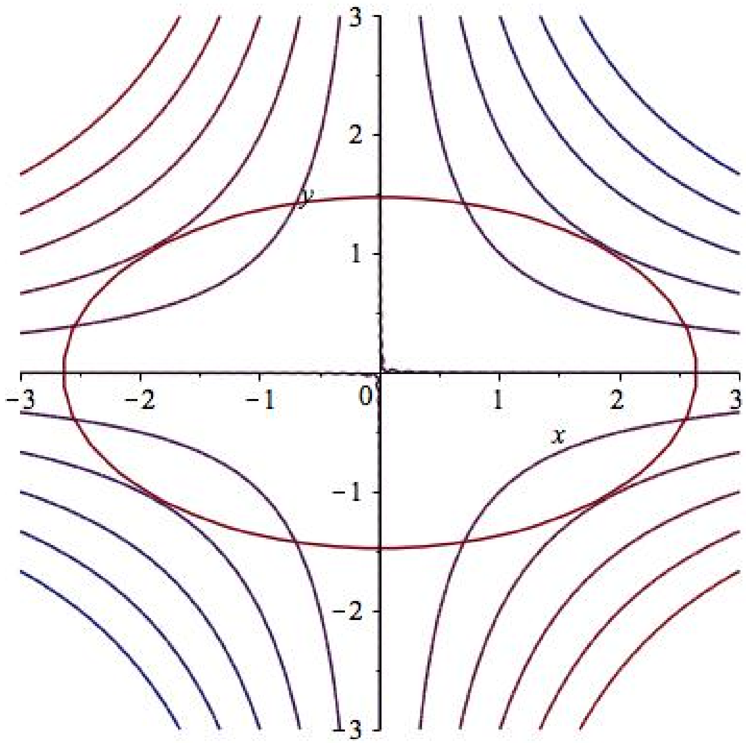
D is *closed* if every edge point of D belongs to D. (In three dimensions, every point on the surface of D belongs to D.)

Definition: The number c is an *absolute maximum* value for $f(x, y)$ on D if there is some point (a, b) in D such that $f(a, b) = c$, and for all points (x, y) in D we have $f(x, y) \leq c$. The absolute maximum value c is attained at (a, b).

The number c is an *absolute minimum* value for $f(x, y)$ on D if there is some point (a, b) in D such that $f(a, b) = c$, and for all points (x, y) in D we have $f(x, y) \geq c$. The absolute minimum value c is attained at (a, b).

Theorem: A continuous function $f(x, y)$ defined on a closed bounded region D has an absolute maximum value and an absolute minimum value on D. The points at which those extreme values are attained are either critical points of f or edge points of D.

Preliminary Homework: Pictured are some level curves of the function $f(x, y) = xy$, and an ellipse γ, which is a level curve of a function $g(x, y)$.



1. Give the approximate coordinates of the points on γ at which $f(x, y)$ is largest and smallest.

2. What relationship do the level curves of f and of g have at those points?

3. What relationship do the gradients of f and of g have at those points? Why?

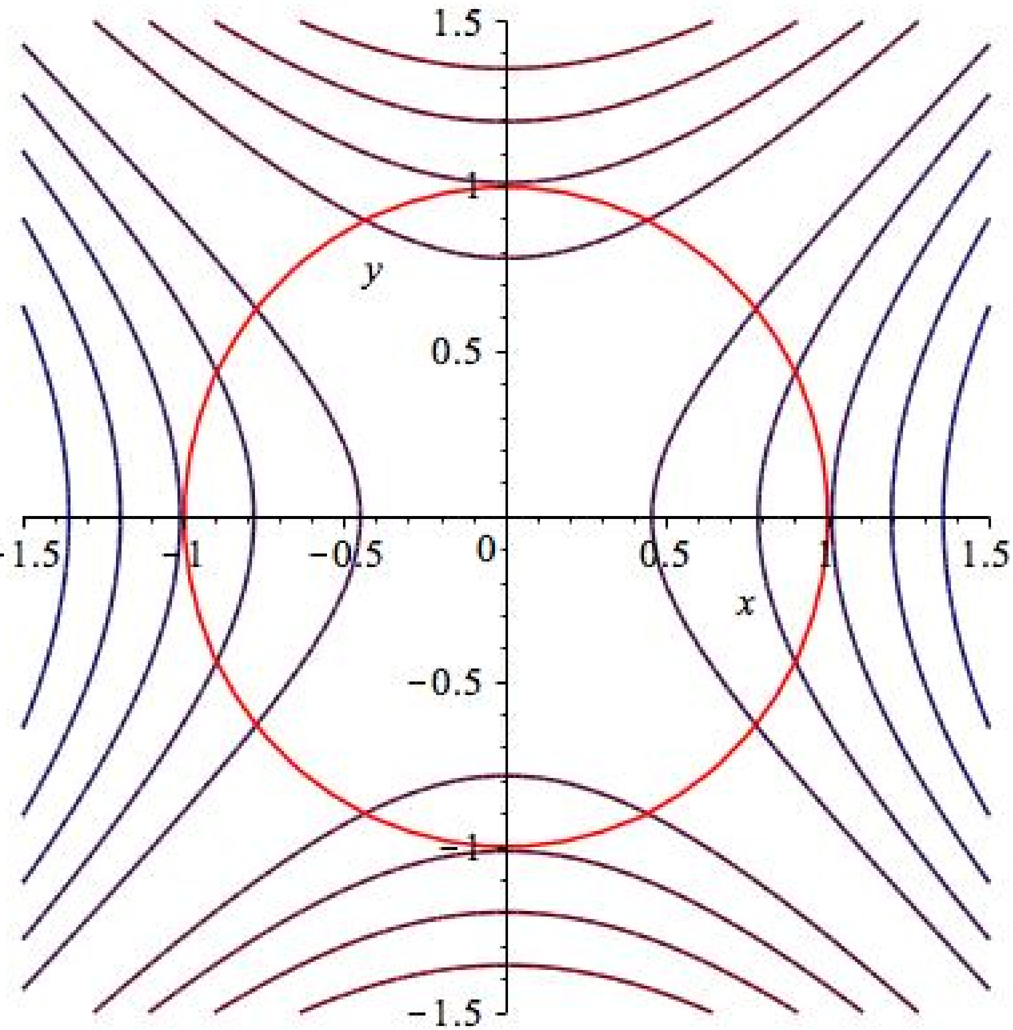
From last class: Find the largest and smallest values of $f(x, y) = x^2 - y^2$ on the region $x^2 + y^2 \le 1$.

There is one critical point of f, the origin $(0, 0)$, and

$$f(0, 0) = 0.$$

This is a possible candidate for the largest or smallest value.

Now we have to check the edge points.



The edge, the circle $x^2 + y^2 = 1$, is a level curve of $g(x, y) = x^2 + y^2$. We look for points at which the circle is tangent to a level curve of $f(x, y) = x^2 - y^2$ by looking for points on the circle at which ∇f and ∇g are parallel. Parallel vectors are scalar multiples of each other. So we want to find points (x, y) (and some scalar λ) satisfying:

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$g(x, y) = 1.$$

For our example, these equations become

$$\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$$

$$x^2 + y^2 = 1.$$

Finding the largest or smallest value of $f(x_1, \ldots, x_1)$ is called an optimization problem. Finding the largest or smallest value of $f(x_1, \ldots, x_n)$ when $(x_1, \ldots, x_n)$ is required to satisfy some condition (for example, $x^2 + y^2 = 1$) is called a constrained optimization problem, and the condition is the constraint.

When we are trying to maximize or minimize f on a closed, bounded, region, looking at the edge of that region generally involves constraints of the form $g(x_1, \ldots, x_n) = k$ (for example, $x^2 + y^2 = 1$). In other words, $(x_1, \ldots, x_n)$ must lie on some level set (level curve, level surface, ...) of g.

The method we just used, called the method of Lagrange multipliers, is designed to solve exactly this kind of problem.

Theorem (the method of Lagrange multipliers): Suppose $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ are differentiable functions, and S is a level set of g, defined by $g(x_1, \ldots, x_n) = k$.

If $f(x_1, \ldots, x_n)$ has a largest (or smallest) value on S, then it attains that extreme value at a point $(x_1, \ldots, x_n)$ at which either

$$\nabla g(x_1, \ldots, x_n) = \vec{0}$$

or, for some scalar λ,

$$\nabla f(x_1, \ldots x_n) = \lambda \nabla g(x_1, \ldots, x_n).$$

This means that to solve this problem, we should look for solutions to

$$\nabla g(x_1, \ldots, x_n) = \vec{0} \quad \& \quad g(x_1, \ldots, x_n) = k$$

and to

$$\nabla f(x_1, \ldots, x_n) = \lambda \nabla g(x_1, \ldots, x_n) \quad \&$$
$$g(x_1, \ldots, x_n) = k.$$

Example: Find the distance between the plane $x+2y-z = 16$ and the point $P = (3, 1, 1)$.

To do this, we find the point on the plane that is closest to P. The distance between any point (x, y, z) and the point P is

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z-1)^2},$$

and we want to find the smallest value of d on the plane. It is easier to find the smallest value of d^2, so we define

$$f(x, y, z) = (x-3)^2 + (y-1)^2 + (z-1)^2.$$

We want to find the smallest value of $f(x, y, z)$ on the surface $x + 2y - z = 16$, which is a level surface of the function $g(x, y, z) = x + 2y - z$.

Note: Since the plane is not a closed, bounded region, it is only by geometry that we know there *is* a point on the plane closest to P. If we were trying to find the point on the plane farthest from P we would go through the same steps, but we would not succeed.

Exercise: Use Lagrange multipliers to find the largest value of xy on the ellipse with equation

$$4x^2 + y^2 = 4.$$

Exercise: Find the largest and smallest values of the function $f(x, y, z) = x^2 - y^2 + z^2$ on the region $x^2 + 4y^2 + 9z^2 \leq 36$.

Remember to check critical points of f inside the region.

Then use the method of Lagrange multipliers to look for maximum and minimum points on the surface of the region.

The surface of the region is the ellipsoid with equation $x^2 + 4y^2 + 9z^2 = 36$, so it is a level surface of the function $g(x, y, z) = x^2 + 4y^2 + 9z^2$.

Exercise: If possible, use the method of Lagrange multipliers to find the largest value of the function $f(x, y) = x^2 + 4y^2$ on the branch of the hyperbola $xy = 1$ in the first quadrant. If this is not possible, explain why.

Now do the same thing for the smallest value of f.