Math 8
Winter 2020
Section 1
January 10, 2020

First, some important points from the last class:

$\Big(a_n\Big)_{n=1}^{\infty}$ denotes the infinite sequence $\Big(a_1, a_2, a_3, \dots\Big)$.

$\lim_{n\to\infty} a_n = L$ is defined to mean: For every $\varepsilon > 0$, there is an N, such that for all $n > N$ we have $|L - a_n| < \varepsilon$.

Important examples are sequences of Taylor polynomials

$$(T_n(x))_{n=0}^{\infty} = \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right)_{n=0}^{\infty}.$$

We hope to be able to show that, in many cases,

$$\lim_{n\to\infty} T_n(x) = f(x).$$

We already know this is the case when $f(x) = \dfrac{1}{1-x}$, the Taylor polynomials are centered at 0, and $|x| < 1$. This is because the Taylor polynomials are given by,

$$T_n(x) = 1 + x + x^2 + \cdots + x^n,$$

which we showed is the same as

$$T_n(x) = \frac{1-x^{n+1}}{1-x},$$

so for $|x| < 1$ we have

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} = f(x).$$

However, for $|x| \geq 1$ we know $\lim_{n\to\infty} T_n(x)$ diverges, so things don't always work out.

Prelmiinary Homework:

In the last preliminary homework, you showed the n^{th} degree Taylor polynomial $T_n(x)$ for the function $f(x) = \dfrac{1}{1-x}$ centered at the point $a = 0$ is $T_n(x) = \displaystyle\sum_{k=0}^{n} x^k$.

We are interested in the limit, for particular values of x, which you may have seen (or can see from the Day 2 notes) is the actual value $\dfrac{1}{1-x}$ if $|x| < 1$, and does not exist if $|x| \geq 1$. We may write this limit as an infinite sum,

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty}\left(\sum_{k=0}^{n} x^k\right) = \begin{cases} \frac{1}{1-x} & \text{if } |x| < 1; \\ \text{undefined} & \text{if } |x| \geq 1. \end{cases}$$

Use this formula to find the following infinite sums.

1. $\displaystyle\sum_{k=0}^{\infty} \frac{1}{3^k}$.

   This is $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$.

2. $\displaystyle\sum_{k=0}^{\infty} \frac{4}{3^k}$. (Hint: You can factor the 4 out of the sum.)

   This is $\displaystyle 4\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = 4\left(\frac{3}{2}\right) = 6$.

3. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{3^k}$. (Be careful; this sum doesn't start at $k = 0$.)

   This is (1) minus its first two terms, $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k - \left(1 + \frac{1}{3}\right) = \frac{3}{2} - \frac{4}{3} = \frac{1}{6}$.

   You can also rewrite the sum using $i = k - 2$ and $k = i + 2$, so as $k : 2 \to \infty$ we have $i : 0 \to \infty$, and

   $\displaystyle\sum_{k=2}^{\infty} \frac{1}{3^k} = \sum_{i=0}^{\infty} \frac{1}{3^{i+2}} = \sum_{i=0}^{\infty} \frac{1}{3^2}\frac{1}{3^i} = \frac{1}{9}\sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{1}{9}\left(\frac{3}{2}\right) = \frac{1}{6}$.

Our major interest in discussing limits of sequences is to find limits of Taylor polynomials,

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

We may write this limit instead as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Definition: An infinite series is a sum of infinitely many terms,

$$\sum_{k=0}^{\infty} a_k.$$

(A sequence is a list; a series is a sum.) The sum is defined as follows: The n^{th} partial sum is the sum of the first n terms,

$$S_n = \sum_{k=0}^{n-1} a_k,$$

and the sum of the series is the limit of the partial sums,

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} a_k.$$

If this limit exists and is a number, the series converges; if not, it diverges.

Definition: The Taylor series for $f(x)$ centered at a is the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This is sort of the degree infinity Taylor polynomial. Its n^{th} partial sum is the degree n Taylor polynomial for $f(x)$ centered at a.

Remark: We hope that the sum of the Taylor series for $f(x)$ is equal to $f(x)$. In many nice cases, it is. Taylor's inequality (Taylor's error formula) can help us figure out when that happens. (We are not covering this in Math 8, but you can read about it in the textbook.)

The series we saw in the preliminary homework are called geometric series.

Definition: A geometric series is a series

$$a_0 + a_0 r + a_0 r^2 + \cdots = \sum_{k=0}^{\infty} a_0 r^k.$$

You can identify a geometric series by the first term a_0 and the ratio $r = \dfrac{a_{n+1}}{a_n}$, which must be the same for every n. We already know how to find their sum.

Proposition: The sum of a geometric series with first term a_0 and ratio r is

$$\sum_{k=0}^{\infty} a_0 r^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_0 r^k = \begin{cases} \dfrac{a_0}{1-r} & \text{if } |r| < 1; \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

Example: Find the sum

$$\sum_{k=0}^{\infty} \frac{2^{k-1}}{5 \cdot 3^{2k+1}}.$$

Taylor Series

Here are some important Maclaurin series (Taylor series centered at 0), and the values of x for which they converge to the functions from which they arise (just a fact we don't yet have the tools to prove):

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad -\infty < x < \infty$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad -\infty < x < \infty$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad -\infty < x < \infty$$

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \qquad -1 \leq x \leq 1$$

$$\ln(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \qquad -1 < x \leq 1$$

Example: The Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

For every x we have

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

In particular, taking $x = 1$, we get

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e^1 = e$$

Example: Use the fact that

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \qquad \text{for} \qquad -1 \leq x \leq 1$$

to express π as the sum of an infinite series of fractions.

Some Rules for Series

There is an extensive theory of sequences and series, most of which we will not see in Math 8. In this section, we state a few rules that should make sense. This is not a collection of facts to memorize. This is reassurance that your common sense conclusions about series and sequences are generally valid.

Generally the limits A and B in these rules are assumed to be numbers. The rules also apply to limits of ∞ and $-\infty$, as long as the expression you are evaluating is defined ($\infty + \infty = \infty$) rather than undefined ($\infty - \infty$ is undefined). Be warned that the quotient $\frac{\infty}{0}$ is undefined, *not* ∞. That is because if a_n approaches ∞ and b_n approaches 0 while oscillating between positive and negative values, then $\frac{a_n}{b_n}$ will also oscillate between positive and negative values, and therefore will not approach ∞.

You are free to use these rules in any homework or exam problem (unless the instructions say otherwise, such as, "use the definition of limit.") You do not have to cite the rule by name, as long as you make clear what fact you are using.

1. (constant multiple rule)

   If c is a constant, then

$$\left(\sum_{n=0}^{\infty} a_n = A\right) \implies \sum_{n=0}^{\infty}(ca_n) = cA.$$

2. (addition and subtraction rules)

$$\left(\sum_{n=0}^{\infty} a_n = A \;\&\; \sum_{n=0}^{\infty} b_n = B\right) \implies \sum_{n=0}^{\infty}(a_n \pm b_n) = A \pm B.$$

3. (tail end rule)

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \sum_{n=k}^{\infty} a_n \text{ converges.}$$

   In fact,

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

4. (comparison rule)

   If $a_n \leq b_n$ for all n, then

$$\left(\sum_{n=0}^{\infty} a_n = A \;\&\; \sum_{n=0}^{\infty} b_n = B\right) \implies A \leq B.$$

5. (decreasing terms rule)

   If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{n\to\infty}(a_k) = 0$.

   The converse of this is false, as you can see from the series

   $$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots ,$$

   which does not converge even though the individual terms do approach zero.

6. (nonnegative series rule)

   If $a_n \geq 0$ for all n, then $\sum_{n=0}^{\infty} a_n$ either converges to a finite sum or approaches $+\infty$.

These rules basically follow from applying sequence rules to the sequences of partial sums. For example, the nonnegative series rule follows from the monotone sequence theorem, since if $a_n \geq 0$ for all n, then the sequence of partial sums is an increasing sequence, which must either converge to a number or diverge to $+\infty$.

Convergence

You may notice that we have listed many more sequence rules than series rules. This is because there are many more elementary methods for finding the limit of a sequence than for finding the sum of a series.

Sometimes, even if we cannot find the sum of a series, we can determine whether the series converges or not. There are a number of different convergence tests for series. We will see a few of them now.

Be warned, the next proposition applies *only* to nonnegative series, which are series with no negative terms.

Proposition (the comparison test): Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are non-negative series. If $0 \leq b_n \leq a_n$ for all n, then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \implies \sum_{n=0}^{\infty} b_n \text{ converges.}$$

This proposition follows from the fact that a series with nonnegative terms must either converge or approach infinity. If the larger series does not approach infinity, the smaller one cannot do so either, so it must converge.

The following definition turns out to be useful.

Definition: The series $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.

The reason it is useful is the following proposition.

Proposition: If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then it is convergent.

This proposition follows from things we have already noted. By the sum rule, we can break up a series into the sum of its positive terms and the sum of its negative terms, and show each of those series converges by comparison to the sum of the absolute values of the terms. (There are more details in the last section.)

Be warned that the converse of this proposition is false. There are some series that are convergent but not absolutely convergent. The alternating harmonic series,

$$\sum_{n=1}^{\infty} \left((-1)^{n+1} \left(\frac{1}{n} \right) \right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots ,$$

is one important example.

We can use this proposition to prove $\sum_{n=0}^{\infty} a_n$ converges by proving $\sum_{n=0}^{\infty} |a_n|$ converges. This is often easier, because we have tests such as the comparison test that apply only to nonnegative series.

Example: The series

$$1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} - \frac{1}{256} + \cdot$$

is not geometric, but we can still show it converges by showing it is absolutely convergent.

To do that, we must show that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \cdot$$

converges. This series is geometric, with $r = \frac{1}{2}$, so it converges.

Note: We don't know what our series converges to, exactly. But we can use the comparison test to show the sum must be somewhere between -2 and 2.

Alternating Series

Here is a convergence test that applies to series that are not non-negative. It applies to alternating series, series whose terms alternate between positive and negative.

For an example of an alternating series, remember that the Maclaurin polynomials (Taylor polynomials with center 0) for $\cos(x)$ have the form

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

so we can write the Maclaurin series for $\cos(x)$ as

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

Whatever value we use for x, the terms of this series alternate between positive and negative.

Proposition (the alternating series test): If a series $\sum_{k=0}^{\infty} a_k$ satisfies the following three conditions, then it converges:

(1.) The terms a_n alternate between positive and negative.

(2.) The terms a_n are decreasing in absolute value, that is, $|a_{n+1}| \leq |a_n|$ for all n.

(3.) The terms a_n are approaching zero, $\lim_{n \to \infty} a_n = 0$.

There is a nice proof, with picture, at the beginning of section 11.5 in the textbook.

Example: Show that the Maclaurin series for $\cos(x)$ converges for $x = 1$.

Example: Show that the Maclaurin series for $\cos(x)$ converges for $x = 10$.

This is a harder problem, because this series doesn't quite satisfy all the conditions of the alternating series test. The series is

$$\sum_{k=0}^{\infty} (-1)^k \frac{10^{2k}}{(2k)!}.$$

The terms do alternate between positive and negative, and they do approach zero, since we have already seen that if c is any constant then $\lim_{n\to\infty} \frac{c^n}{n!} = 0$. However, the first few terms are

$$1, -\frac{100}{2}, \frac{10,000}{24}, \ldots,$$

which are clearly not getting smaller in absolute value.

Exercise: Find $\sum_{k=2}^{\infty} \left(\frac{2^k + 3^{k-1}}{5^k} \right)$.

Hint: Break this up as the sum of two sequences. Warning: Notice the lower limit on k.