Math 8
Winter 2020
Section 1
January 13, 2020

First, some important points from the last class:

Definition: An infinite series is a sum of infinitely many terms,

$$\sum_{k=0}^{\infty} a_k.$$

The sum of the series is the limit of the partial sums,

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} a_k.$$

Definition: The Taylor series for $f(x)$ centered at a is the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Here are some important Maclaurin series (Taylor series centered at 0), and the values of x for which they converge to the functions from which they arise (just a fact we don't yet have the tools to prove):

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad -\infty < x < \infty$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad -\infty < x < \infty$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad -\infty < x < \infty$$

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \qquad -1 \le x \le 1$$

$$\ln(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \qquad -1 < x \le 1$$

Some rules for series: constant multiple rule, addition and subtraction rules, tail end rule, comparison rule, decreasing terms rule, nonnegative series rule.

Proposition (the comparison test): Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are non-negative series. If $0 \le b_n \le a_n$ for all n, then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \implies \sum_{n=0}^{\infty} b_n \text{ converges.}$$

Definition: The series $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Proposition: If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Proposition (the alternating series test): If a series $\sum_{k=0}^{\infty} a_k$ satisfies the following three conditions, then it converges:

(1.) The terms a_n alternate between positive and negative.
(2.) The terms a_n are decreasing in absolute value, that is, $|a_{n+1}| \le |a_n|$ for all n.
(3.) The terms a_n are approaching zero, $\lim_{n\to\infty} a_n = 0$.

We now have two general ways of finding the sum of a series:

1. Recognize the series as a geometric series, and use the formula for the sum of a geometric series;

2. Recognize the series as a the Taylor series (that we know converges to the value of the function) with some particular value of x.

Example: Find the sum of the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}.$$

Since this series has terms of alternating sign, and even factorials in the denominators, we compare it to the Taylor series for $\cos(x)$:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Substituting $x = 1$ gives our series:

$$\cos(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}.$$

We can use this to find an approximate value for $\cos(1)$, by taking a partial sum:

$$\cos(1) = \sum_{k=0}^{3} \frac{(-1)^k}{(2k)!} = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} = .54027778\ldots.$$

However, this approximation isn't correct to that many decimal places. By calcuator,

$$\cos(1) = 0.54030230\ldots$$

Here we wanted to estimate the value of an infinite sum, and we used a partial sum as an approximation. Today we will learn ways to figure out how far off our approximation could be.

Definition: The error in using a partial sum S_n as an approximation to the actual sum S of an infinite series is the difference between them,

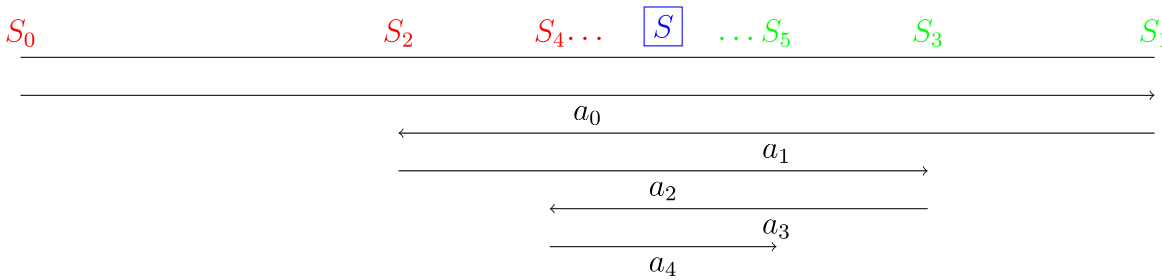
$$error = |S - S_n|.$$

A number b is a bound for the error if

$$error \leq b.$$

"Bounding the error" means finding a bound for the error.

Alternating Series Error Bound: Suppose we have a series $\sum_{k=0}^{\infty} a_k$ that meets the conditions of the alternating series test. We saw this series must converge by looking at the following picture:



Here we see the even-numbered partial sums increasing toward the actual sum S, and the odd-numbered partial sums decreasing toward S. In particular, we see that the sum S is less than the first term a_0.

We also see that, for every n, the sum S is between the partial sum S_n and the next partial sum S_{n+1}. Therefore, the error in using the partial sum S_n as an approximation for S is the difference between S_n and S_{n+1}:

$$error = |S - S_n| \leq |S_{n+1} - S_n| = a_n.$$

Rather than worrying about subscripts, just remember the first term you do NOT include is a bound for the error.

Example: Find a bound for the error in approximating $\cos(1) = 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \cdots$ with the fourth partial sum $1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!}$.

Since this series satisfies the alternating series test conditions, the error is at most the absolute value of the first term not included,

$$error \leq \frac{1}{8!} = \frac{1}{40320} = .00002480\ldots.$$

Example: Recall that we can use the Maclaurin series for $\tan^{-1}$ to express π as an infinite series:

$$\pi = 4\tan^{-1}(1) = 4\sum_{k=0}^{\infty}\frac{(-1)^k(1)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k(4)}{2k+1}.$$

Find a partial sum that gives π correctly to 4 decimal places (error at most .00005).

We want our error to be at most $\frac{1}{20,000}$. Since this series satisfies the alternating series test, the error is at most the absolute value of the next term. The absolute value of the k^{th} term is $\frac{4}{2k+1}$, so we want

$$\frac{4}{2k+1} \le \frac{1}{20,000} \qquad 2k+1 \ge 80,000 \qquad k \ge 39,999.5$$

so to be guaranteed this accuracy, we want the first term left out to be $k = 40,000$:

$$\pi \approx \sum_{k=0}^{39{,}999}\frac{(-1)^k(4)}{2k+1}.$$

Exercise: Write down an infinite sum that equals $\sin(1)$. Find a partial sum that approximates $\sin(1)$ correctly to within three decimal places.

Comparison Test Bounds:

Example: Use the Maclaurin series for $\ln(x+1)$ to approximate $\ln(.9)$ with an error of at most .001.

$$\ln(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \qquad -1 < x \leq 1$$

so we can substitute $x = -.1$ to get

$$\ln(.9) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-.1)^k}{k} = \sum_{k=1}^{\infty} \frac{-1}{k(10)^k}$$

The error in using the partial sum S_n to approximate $\ln(.9)$ is the sum of the terms we did not include:

$$|\ln(.9) - S_n| = \left| \sum_{k=1}^{\infty} \frac{-1}{k(10)^k} - \sum_{k=1}^{n} \frac{-1}{k(10)^k} \right| = \sum_{k=n+1}^{\infty} \frac{1}{k(10)^k}.$$

We can compare this to a geometric series

$$\sum_{k=n+1}^{\infty} \frac{1}{k(10)^k} \leq \sum_{k=n+1}^{\infty} \frac{1}{(n+1)(10)^k} = \frac{\frac{1}{(n+1)(10)^{n+1}}}{1-\frac{1}{10}} = \frac{1}{(n+1)(10)^{n+1}} \frac{10}{9} = \frac{1}{9(n+1)(10)^{n+1}}$$

We want this error to be at most .001, so we want

$$\frac{1}{9(n+1)(10)^{n+1}} \leq \frac{1}{10^3} \qquad 9(n+1)(10)^{n+1} \geq 10^3.$$

Here $n = 2$ will do. Our approximation is

$$\ln(.9) \approx \sum_{k=1}^{2} \frac{-1}{k(10)^k} = -\frac{21}{200} = -.105.$$

By calculator, $\ln(.9) = -.10536\ldots.$

Exercise: Approximate e to within 2 decimal places. Hint: $e = e^1$, and we can express e^x as a Taylor series.

Some examples from last class:

Example: Show that the Maclaurin series for $\cos(x)$ converges for $x = 1$.

This series, we see by substituting $x = 1$ into the Maclaurin series we had above, is

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!}.$$

It is easy to see that all three conditions of the alternating series test are satisfied.

Example: Show that the Maclaurin series for $\cos(x)$ converges for $x = 10$.

This is a harder problem, because this series doesn't quite satisfy all the conditions of the alternating series test. The series is

$$\sum_{k=0}^{\infty} (-1)^k \frac{10^{2k}}{(2k)!}.$$

The terms do alternate between positive and negative, and they do approach zero, since we have already seen that if c is any constant then $\lim_{n\to\infty} \frac{c^n}{n!} = 0$. However, the first few terms are

$$1, -\frac{100}{2}, \frac{10,000}{24}, \ldots,$$

which are clearly not getting smaller in absolute value.

From the tail end test, it is enough to show that the tail end

$$\sum_{k=5}^{\infty} (-1)^k \frac{10^{2k}}{(2k)!}$$

converges. For this series, we have $a_k = (-1)^k \frac{10^{2k}}{(2k)!}$, and we can write

$$|a_{k+1}| = \frac{10^{2k+2}}{(2k+2)!} = \left(\frac{10 \cdot 10 \cdot 10^{2k}}{(2k+2)(2k+1)(2k)!} \right) = \left(\frac{10}{2k+2} \right) \left(\frac{10}{2k+1} \right) a_k,$$

which is less than a_k, because $k \geq 5$ and so $\frac{10}{2k+2} < 1$ and $\frac{10}{2k+1} < 1$.

Exercise: Find $\displaystyle\sum_{k=2}^{\infty} \left(\frac{2^k + 3^{k-1}}{5^k} \right)$.

We can break this up into the sum of two series, $\sum_{k=2}^{\infty}\left(\frac{2^k}{5^k}\right)$, which is a geometric series with first term $\frac{4}{25}$ and ratio $\frac{2}{5}$, and $\sum_{k=2}^{\infty}\left(\frac{3^{k-1}}{5^k}\right)$, which is a geometric series with first term $\frac{3}{25}$ and ratio $\frac{3}{5}$. Both ratios have absolute value less than one, so the sum is

$$\frac{\frac{4}{25}}{1-\left(\frac{2}{5}\right)}+\frac{\frac{3}{25}}{1-\left(\frac{3}{5}\right)}=\frac{\frac{4}{25}}{\left(\frac{3}{5}\right)}+\frac{\frac{3}{25}}{\left(\frac{2}{5}\right)}=\frac{1}{5}\left(\frac{4}{3}+\frac{3}{2}\right)=\frac{17}{30}.$$