Math 8
Winter 2020
Section 1
January 15, 2020

First, the takeaway from last class:

Definition: The error in using a partial sum S_n as an approximation to the actual sum S of an infinite series is the difference between them,

$$error = |S - S_n|\,.$$

A number b is a bound for the error if

$$error \leq b.$$

"Bounding the error" means finding a bound for the error.

Alternating Series Error Bound: Suppose we have a series $\sum_{k=0}^{\infty} a_k$ that meets the conditions of the alternating series test. Then, the error in using the partial sum $S_n = \sum_{k=0}^{n-1} a_k$ as an approximation for S is bounded by the difference between S_n and S_{n+1}, which is the last term of S_{n+1}:

$$error = |S - S_n| \leq |S_{n+1} - S_n| = \left| \sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k \right| = |a_n|.$$

Rather than worrying about subscripts, just remember the first term you do NOT include is a bound for the error.

Comparison Test Error Bound: Since the difference

$$S - S_n = \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{n-1} a_k = \sum_{k=n}^{\infty} a_k$$

is itself a series, we can also find a bound by using the comparison rule:

If $a_n \leq b_n$ for all n, then

$$\left(\sum_{n=0}^{\infty} a_n = A \ \&\ \sum_{n=0}^{\infty} b_n = B \right) \implies A \leq B.$$

Here is one more convergence test.

Proposition: (the ratio test) For any series, if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

then

$$\sum_{n=0}^{\infty} a_n \text{ is } \begin{cases} \text{absolutely convergent} & \text{if } L < 1; \\ \text{divergent} & \text{if } L > 1; \\ \text{we cannot tell from this test} & \text{if } L = 1. \end{cases}$$

Example: Show the Maclaurin series for e^x,

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for all x.

Let x be any particular number. (So x is a constant.) To apply the ratio test to this series, we must look at the limit of (the absolute value of) the ratio of successive terms,

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^n(x)}{n!(n+1)}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Since the limit is less than 1, the series converges.

Radius of Convergence

Taylor series centered at $x = a$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

are examples of power series centered at $x = a$

$$\sum_{k=0}^{\infty} c_k(x-a)^k \qquad \text{(each } c_k \text{ is a constant).}$$

We can use a power series to define a function,

$$g(x) = \sum_{k=0}^{\infty} c_k(x-a)^k,$$

whose domain is the set of x for which the power series converges.

Definition: The *radius of convergence* of the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ is R, where $0 \leq R \leq \infty$, if the power series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$.

(It may or may not converge for $|x-a| = R$.)

Proposition: Every power series has a radius of convergence.

Example: The geometric series $\sum_{k=0}^{\infty} x^k$ converges for $|x| < 1$ and diverges for $|x| > 1$, so its radius of convergence is $R = 1$.

Theorem: Suppose the function $f(x)$ is defined by a power series centered at a with radius of convergence R, meaning that for $|x - a| < R$ we have

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k.$$

Then that power series is the Taylor series for $f(x)$ centered at a.

Example: Because the geometric series $\sum_{k=0}^{\infty} x^k$ converges to $\frac{1}{1-x}$ for $|x| < 1$ and diverges for $|x| > 1$, we know it must be the Maclaurin series for the function $f(x) = \frac{1}{1-x}$. (We can also check this by using the formula for a Taylor series.)

It may be that there is no power series expansion centered at a that converges to $f(x)$ near a.

This must mean that the Taylor series for $f(x)$ centered at $x = a$ does not converge to $f(x)$ in any interval around a. In fact, there are cases in which the Taylor series for $f(x)$ converges to another function entirely.

Example: Define

$$f(x) = \begin{cases} e^{-(x^{-2})} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

This function has derivatives of all orders at 0; in fact, for every n, we have

$$f^{(n)}(0) = 0.$$

Therefore, the Maclaurin series for $f(x)$ is

$$\sum_{k=0}^{\infty} \left(0 \cdot \frac{x^k}{k!} \right),$$

which converges for every x, but (except at $x = 0$) it does not converge to $f(x)$.

Example: A simpler example is $f(x) = |x| = (x^2)^{\frac{1}{2}}$. Its Taylor series about $a = 1$ is

$$1 + (x-1) + 0(x-1)^2 + 0(x-1)^3 + 0(x-1)^4 + 0(x-1)^5 + \cdots = x,$$

which converges for all x, but converges to $f(x)$ only for $x \geq 0$.

We can often use the ratio test to find the radius of convergence.

Example: Find the radius of convergence of the series $\sum_{k=0}^{\infty} \frac{x^k}{k^2}$. (Notice that we can see immediately the center is 0.)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}} \right| = \lim_{n\to\infty} \left| x \frac{n^2}{(n+1)^2} \right| = |x|.$$

By the ratio test, this converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

The radius of convergence is 1.

Example: Find the radius of convergence of the power series $\sum_{k=0}^{\infty} k(x-1)^k$. For what values of x do we know this series converges?

New Taylor Series from Old

If we compute the Taylor series for $f(x)$ centered at a directly from the formula for Taylor series, we can use the ratio test to find the radius of convergence. For x within that radius of convergence of a, we hope that the Taylor series for $f(x)$ not only converges, but converges to $f(x)$. The ratio test cannot tell us that.

However, there is another way to arrive at Taylor series, and know they converge to the function.

Theorem: Suppose the function $f(x)$ is defined by a power series centered at a with radius of convergence R,

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k.$$

Then the term-by-term derivative of the power series has the same radius of convergence, and gives the derivative of $f(x)$:

$$f'(x) = \sum_{k=0}^{\infty} k c_k (x-a)^{k-1}.$$

The same is true for the term-by-term indefinite integral,

$$\int f(x)\,dx = C + \sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1},$$

and definite integral, as long as b and d are in $(a-R, a+R)$,

$$\int_b^d f(x)\,dx = \left[\sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1}\right]\Bigg|_{x=b}^{x=d} = \sum_{k=0}^{\infty} \left(\left[c_k \frac{(x-a)^{k+1}}{k+1} \right]\Bigg|_{x=b}^{x=d} \right).$$

In particular, for $a - R < x < a + R$ (inside the radius of convergence)

$$\int_a^x f(u)\,du = \left[\sum_{k=0}^{\infty} c_k \frac{(u-a)^{k+1}}{k+1}\right]\Bigg|_{u=a}^{u=x} = \sum_{k=0}^{\infty} \left(\left[c_k \frac{(x-a)^{k+1}}{k+1} \right] \right).$$

Example: It is possible to show that for all x (in other words, with radius of convergence $R = \infty$), the Maclaurin series for $\sin(x)$ converges to $\sin(x)$:

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{(2k+1)}}{(2k+1)!} \right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

By this theorem we can take derivatives of each side, differentiating the power series term-by-term, to get

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{(2k+1)x^{2k}}{(2k+1)!} \right) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k}}{(2k)!} \right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$

with the same radius of convergence $R = \infty$. This shows the Maclaurin series for $\cos(x)$ also converges to $\cos(x)$ for every x.

We can also integrate series term-by-term. Since

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

we can take an antiderivative (remembering the constant of integration) to get

$$-\cos(x) = C + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} = \frac{x^8}{8!} + \cdots$$

Substituting $x = 0$ gives $-1 = C$, so we get

$$-\cos(x) = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} = \frac{x^8}{8!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Example: We have seen that for $|x| < 1$ we have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Substituting $x = -u^2$, we get

$$\frac{1}{1+u^2} = \sum_{k=0}^{\infty} (-u^2)^k = \sum_{k=0}^{\infty} (-1)^k u^{2k}.$$

This holds for $|-u^2| < 1$, which is to say, $|u| < 1$. Applying our theorem, we get

$$\int_0^x \frac{1}{1+u^2}\,du = \int_0^x \left(\sum_{k=0}^{\infty} (-1)^k u^{2k} \right) dx = \sum_{k=0}^{\infty} \left(\int_0^x (-1)^k u^{2k}\,du \right);$$

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

By the theorem, this has the same radius of convergence, namely 1, so this is true for $|x| < 1$.

The theorem doesn't tell us whether this is true for $|x| = 1$, that is, for $x = 1$ or for $x = -1$. It turns out that it is true in both cases, so the interval of convergence for this series is $[-1, 1]$, and it converges to $\arctan(x)$ for every x in that interval.

Exercise: For $|u| \leq 1$, we know

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k.$$

Integrate both sides from 0 to x.

This should allow you to find a power series expansion for $\ln(1-x)$. For what values of x do we know this holds?

Random Note: Because we can differentiate power series term-by-term, we can use Taylor series to solve differential equations. A simple example of a differential equation is

$$\frac{dy}{dx} = y,$$

where y is a function of x. This equation says that y is its own derivative. To solve this using Taylor series, we assume we can express y as a Taylor series

$$y = \sum_{k=0}^{\infty} c_k x^k,$$

and then use the equation

$$\frac{dy}{dx} = y,$$

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} c_k x^k\right) = \sum_{k=0}^{\infty} c_k x^k$$

For the next step, the derivative of a constant is zero, which is why we start with $k = 1$,

$$\sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=0}^{\infty} c_k x^k$$

For the next step, we rewrite the sum on the left using $i = k - 1$,

$$\sum_{i=0}^{\infty} (i+1) c_{i+1} x^i = \sum_{k=0}^{\infty} c_k x^k$$

Now we can use k in place of i

$$\sum_{k=0}^{\infty} (k+1) c_{k+1} x^k = \sum_{k=0}^{\infty} c_k x^k$$

The coefficients of each x^k must be equal:

$$(k+1)c_{k+1} = c_k \qquad c_{k+1} = \frac{c_k}{k+1}$$

$$c_1 = \frac{c_0}{1} \quad c_2 = \frac{c_1}{2} = \frac{c_0}{2} \quad c_3 = \frac{c_2}{3} = \frac{c_0}{3 \cdot 2} \quad c_4 = \frac{c_3}{4} = \frac{c_0}{4 \cdot 3 \cdot 2} \quad c_k = \frac{c_0}{k!}$$

$$y = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{c_0}{k!} x^k = c_0 \sum_{k=0}^{\infty} \frac{x^k}{k!} = c_0 e^x.$$

That is, the functions that are their own derivatives (and can be written as Taylor series) are the constant multiples of e^x.