Math 8
Winter 2020
Section 1
January 17, 2020

First, the takeaway from last class:

A power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ always has a radius of convergence R with $0 \le R \le \infty$; it converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$.

We can find the radius of convergence of a power series using the ratio test: If

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L,$$

then

$$\sum_{k=0}^{\infty} a_k \begin{cases} \text{converges absolutely} & \text{if } L < 1; \\ \text{diverges} & \text{if } L > 1; \\ \text{cannot tell from ratio test} & \text{if } L = 1. \end{cases}$$

Theorem: Suppose the function $f(x)$ is defined by a power series centered at a with radius of convergence R,

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k.$$

Then the term-by-term derivative of the power series has the same radius of convergence, and gives the derivative of $f(x)$:

$$f'(x) = \sum_{k=0}^{\infty} kc_k(x-a)^{k-1}.$$

The same is true for the term-by-term indefinite integral,

$$\int f(x)\,dx = C + \sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1},$$

and definite integral, as long as b and d are in $(a-R, a+R)$,

$$\int_b^d f(x)\,dx = \left[\sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1}\right]\Bigg|_{x=b}^{x=d} = \sum_{k=0}^{\infty}\left(\left[c_k \frac{(x-a)^{k+1}}{k+1}\right]\Bigg|_{x=b}^{x=d}\right).$$

In particular, for $a-R<x<a+R$ (inside the radius of convergence)

$$\int_a^x f(u)\,du = \left[\sum_{k=0}^{\infty} c_k \frac{(u-a)^{k+1}}{k+1}\right]\Bigg|_{u=a}^{u=x} = \sum_{k=0}^{\infty}\left(\left[c_k \frac{(x-a)^{k+1}}{k+1}\right]\right).$$

This gives us a way to get new Taylor series from old.

Preliminary Homework

Recall that you first defined the definite integral as the limit of Riemann sums. In the simplest form: Break the interval $[a, b]$ into n-many subintervals of length $\Delta x = \dfrac{b-a}{n}$. For each i between 1 and n, choose a point x_i^* in the i^{th} subinterval. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) = \lim_{\Delta x\to 0}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right).$$

(In a less simple form, the subintervals do not have to be the same length, as long as the length of the longest subinterval approaches zero.)

The left endpoint or right endpoint Riemann sum is obtained by choosing x_i^* to be the left or right endpoint of the i^{th} subinterval.

1. Write out the left endpoint Riemann sum with $n = 4$ for the integral

$$\int_1^2 \frac{1}{x}\,dx.$$

You can leave your answer as a sum of fractions.

Endpoints of subintervals:

$$1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2.$$

Length of subintervals: $\Delta x = \dfrac{2-1}{4} = \dfrac{1}{4}$ Choose the left endpoint of each interval:

$$x_1^* = 1,\ x_2^* = \frac{5}{4},\ x_3^* = \frac{3}{2},\ x_4^* = \frac{7}{4}.$$

$$\sum_{i=1}^{4} f(x_i^*)\,\Delta x = \sum_{i=1}^{4}\left(\frac{1}{x_i^*}\right)\left(\frac{1}{4}\right) =$$

$$(1)\left(\frac{1}{4}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{4}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{4}{7}\right)\left(\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}.$$

2. Choose the correct answer. $\lim_{n\to\infty}\left(\sum_{i=1}^{n}(f(x_i^*))^2\Delta x\right) =$

(a) $\left(\int_a^b f(x)\,dx\right)^2$.

(b) $\boxed{\int_a^b (f(x))^2\,dx}$.

(c) Neither (a) nor (b).

Example: An abstract sculpture is 10 feet high, and its horizontal cross-section x feet above its base is a circle of radius $(12 - x + \sin x)$ feet. Find its volume.

We imagine a vertical x-axis, with $x = 0$ at the base of the sculpture, and $x = 10$ at the highest point.

If we slice the sculpture horizontally at height x and look at the resulting cross-section, it has area

$$A(x) = \pi(12 - x + \sin x)^2.$$

We approximate the volume by (mentally) slicing up the sculpture into n-many small horizontal slices of height Δx, and letting V_i be the volume of the i^{th} slice.

We approximate V_i by choosing a height x_i^* in the i^{th} slice, pretending that the cross-section throughout that entire slice is the same as the cross-section at height x_i^*, which has area $A(x_i^*)$, and computing the volume of the slice as cross-sectional area times height. If Δx is small enough, the cross-section shouldn't change very much throughout this thin slice, so this should be a good approximation:

$$A(x_i^*) = \pi(12 - x_i^* + \sin x_i^*)^2;$$

$$V_i \approx A(x_i^*)\,\Delta x = \left(\pi(12 - x_i^* + \sin x_i^*)^2\right)\,\Delta x.$$

The total volume, from the base $x = 0$ to the top $x = 10$ is the sum of the volumes of the small slices, which we have approximated:

$$Vol = \sum_{i=1}^{n} V_i \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \sum_{i=1}^{n} \pi(12 - x_i^* + \sin x_i^*)^2 \Delta x.$$

Now we say that we can get as close an approximation to the volume as we want by taking a large enough number of tiny slices. That is,

$$Vol = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x;$$

$$Vol = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \pi(12 - x_i^* + \sin x_i^*)^2 \Delta x \right).$$

Recognizing this as a limit of Riemann sums, we write

$$Vol = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_0^{10} A(x)\,dx;$$

$$Vol = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \pi(12 - x_i^* + \sin x_i^*)^2 \Delta x \right) = \int_0^{10} \pi(12 - x + \sin x)^2\,dx.$$

We can use this same method to find the volume of any solid. View the solid as laid out along the x-axis from $x = a$ to $x = b$, take cross-sections perpendicular to the x-axis[1], and let $A(x)$ be the cross-sectional area at x. Then

$$\boxed{Volume = \int_a^b A(x)\, dx}.$$

This technique is called Volumes by Slicing.

Integrals can be used to approximate many different things, of which we will see a number of examples. The general idea will always be the same:

1. View your problem as somehow laid out along the x-axis from $x = a$ to $x = b$. The variable x could represent time, or distance along a line, or some other quantity.

2. Divide the interval $a \leq x \leq b$ into n small intervals of length Δx, breaking your problem up into n-many small pieces.

3. Approximate the answer to the i^{th} small piece of your problem. Look for an approximation in the form of $f(x_i^*)\Delta x$, where x_i^* is in your i^{th} subinterval, and $f(x)$ can be any function.

4. Approximate the answer to your problem as a sum of these approximations,

$$Q \approx \sum_{i=1}^{n} f(x_i^*)\, \Delta x.$$

5. Notice that your approximation is a Riemann sum, and take a limit:

$$Q = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x = \int_a^b f(x)\, dx.$$

We will expect you to learn how to do this, so that when you encounter a problem that can be solved by integration, you can figure out how to do it.

[1] It is important that the cross-sections be *perpendicular to your axis.*

Example: The temperature at time t hours after midnight on January 1, 2016, in degrees Fahrenheit, is given by the function $f(t)$. Find the average temperature between times $t = a$ and $t = b$.

We use the same idea of approximating the average and then taking the limit of better and better approximations.

To approximate the average, we can break our time interval into n-many small intervals of length Δt, record the temperature once during each interval, and then take the average of our measurements. If t_i^* is the time during the i^{th} small interval at which we record the temperature, then our measurements are

$$f(t_1^*), f(t_2^*), f(t_3^*), \ldots, f(t_n^*), \text{ or in sequence notation}$$

$$\big(f(t_i^*)\big)_{i=1}^n.$$

The average of these measurements is

$$\frac{\displaystyle\sum_{i=1}^n f(t_i^*)}{n}.$$

The average temperature is the limit as the number of measurements approaches infinity:

$$A = \lim_{n\to\infty} \frac{\displaystyle\sum_{i=1}^n f(t_i^*)}{n} = \lim_{n\to\infty} \left(\sum_{i=1}^n f(t_i^*) \frac{1}{n} \right) =$$

$$\lim_{n\to\infty} \left(\frac{1}{b-a} \sum_{i=1}^n f(t_i^*) \frac{b-a}{n} \right) =$$

$$\left(\frac{1}{b-a} \right) \lim_{n\to\infty} \left(\sum_{i=1}^n f(t_i^*)\, \Delta t \right) = \left(\frac{1}{b-a} \right) \int_a^b f(t)\, dt.$$

Formula: The average value of the function $f(x)$ on the interval $a \leq x \leq b$ is given by

$$\boxed{\text{AVERAGE } = \left(\frac{1}{b-a} \right) \int_a^b f(x)\, dx.}$$

Exercise: A wire of varying composition occupies the portion of the x-axis for $a \leq x \leq b$. The part of the wire at point x has mass density $f(x)$. If we were using the usual SI units, then the units of x would be meters, and the units of $f(x)$ would be kilograms per meter.

If the mass density of the wire were constant, we could compute the total mass of the wire by multiplying the length of the wire (in meters) times the mass density (in kilograms per meter) to find the total mass (in kilograms). If the mass density is not constant, we can still find the total mass.

Follow the following steps to find a formula for the total mass of the wire.

Break the interval $a \leq x \leq b$ up into n-many small subintervals of length Δx, each of which is occupied by a small piece of the wire.

Choose a point x_i^* in the i^{th} small subinterval, and approximate the mass of the i^{th} small piece of wire by pretending the mass density throughout that piece is the same as the mass density at point x_i^*. If Δx is small enough, the mass density shouldn't change very much over that distance, and this should be a good approximation.

$$\text{Mass density of } i^{th} \text{ piece} \approx \underline{\hspace{6em}}$$

$$\text{Length of } i^{th} \text{ piece} = \underline{\hspace{6em}}$$

$$\text{Mass of } i^{th} \text{ piece} = m_i \approx \underline{\hspace{6em}}$$

Now approximate the total mass by adding up the approximate masses of the small pieces.

$$\text{Mass} = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} \underline{\hspace{6em}}$$

Finally, take a limit as the number of pieces approaches infinity. This should give you an integral.

$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} \underline{\hspace{6em}} =$$

$$\int_{\boxed{}}^{\boxed{}} \underline{\hspace{6em}}\, d\underline{\hspace{3em}}$$

Exercise: The mass density of a wire occupying the portion of the x-axis for $0 \le x \le 4\pi$ is $f(x) = \cos^3(x)\sin^3(x) + 2$.

Find the total mass, using your answer to the preceding problem.

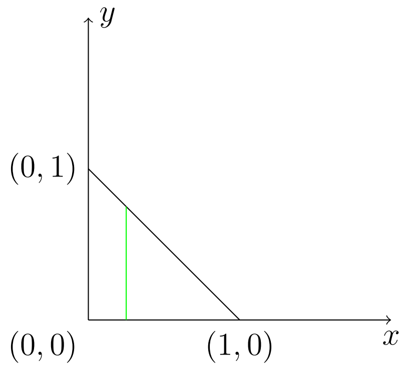
Find the average mass density, using the formula for the average value of a function over an interval.

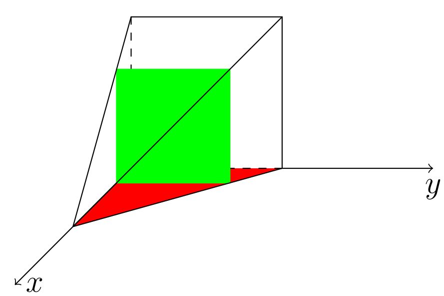
Notice that the total mass is the average mass density times the length of the wire. Will this always happen? Why?

Exercise: The base of a given solid is the triangle with corners $(0,0)$, $(1,0)$, and $(0,1)$, and the cross-sections perpendicular to the x-axis are squares. (You should imagine that the xy-plane is horizontal, and we are taking vertical slices perpendicular to the x-axis, as in the picture below.) Find the volume of this solid.

The first picture shows the base of the solid. The green line is the bottom edge of one slice perpendicular to the x-axis.

In the second picture, the base of the solid is colored red, and one slice perpendicular to the x-axis is colored green.





Exercise: Find the average value of the functions $f(x) = \cos(x)$ and $g(x) = \cos^2(x)$ on the interval $0 \leq x \leq \pi$.

Challenge Problem: Follow these steps to find a formula for the length of the curve $y = f(x)$ for $a \leq x \leq b$. We will assume that $f'(x)$ exists and is continuous over this interval.

Drawing pictures will be useful for this problem.

First solve this problem: A right triangle in the xy-plane has a horizontal leg of length Δx, a vertical leg of length Δy, and a hypotenuse of slope m. Find the length of the hypotenuse in terms of m and Δx.

You should be able to write your answer in the form (____________)Δx, where m appears in the expression inside the parentheses, but Δx does not.

Now approximate the length of the curve as follows:

Break the x-axis interval $a \leq x \leq b$ up into n-many small x-axis intervals of length Δx. This breaks the curve up into n-many small pieces, one lying above each small x-axis interval.

Choose a point x_i^* in each small x-axis interval.

Approximate the length of the i^{th} small piece of the curve by pretending it is a line segment with slope $f'(x_i^*)$ lying above the i^{th} small x-axis interval. If Δx is small enough, the slope won't change very much over this small interval, so this will be a good approximation. (When we zoom in enough on the graph of a differentiable function, it looks a lot like a straight line.)

Use your answer to the "first solve this problem" problem to find the approximation.

Approximate the length of the entire curve by adding up the approximate lengths of the small pieces.

Now take a limit as $n \to \infty$, and express that limit as an integral.

You can test your answer by applying it to the function $f(x) = \dfrac{4x}{3}$ and the interval $0 \leq x \leq 3$. (Use geometry to see what your answer should be.)