Math 8
Winter 2020
Section 1
January 22, 2020

First, some important points from the last class:

Many applications of integration can be found in the following way:

1. Break the problem into many tiny pieces.
2. Approximate the answer for each piece by pretending some relevant function is constant on that piece.
3. Take the sum of these approximate answers, and write it in the form of a Riemann sum.
4. Take the limit as the number of pieces approaches ∞, and get your answer in the form of an integral.

We found three formulas this way:

Volumes by slicing: If a solid lies along the portion of the x-axis $a \le x \le b$, and its cross-sectional area perpendicular to the x-axis at point x is $A(x)$, then its volume is

$$\int_a^b A(x)\,dx.$$

The average value of the function $f(x)$ for $a \le x \le b$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

If a wire (for example) can be laid out on the x-axis interval $a \le x \le b$, and the mass density of the wire at point x is $f(x)$, then the total mass of the wire is

$$\int_a^b f(x)\,dx.$$

This same idea can be used, for example, to find total charge from charge density.

Preliminary Homework: In these problems, we will find the formula for the volume of a cone of height h and base radius r.

(1.) If you slice the cone with a plane parallel to its base at a distance of x units below the top point, that slice is a disc, and the portion of the cone above the plane is a smaller cone.

The height of the smaller cone is x. What is its base radius? $\boxed{\frac{rx}{h}}$.

(2.) Draw the x-axis passing through the point of the cone and the center of its base, with $x = 0$ at the top of the cone and $x = h$ at the base of the cone. Break the interval $[0, h]$ into n-many subintervals of length Δx, and slice the cone with planes parallel to the base of the cone at each division point of the subintervals. This slices the cone into a bunch of slabs, one for each subinterval.

Choose x_i^* in the i^{th} subinterval. The i^{th} slab is almost the shape of a coin (a really, really short cylinder) with:

(a.) Thickness equal to $\boxed{\Delta x}$

(b.) Base radius equal to $\boxed{\frac{rx_i^*}{h}}$;

(c.) Volume equal to $\boxed{\frac{\pi r^2 (x_i^*)^2}{h^2}\,\Delta x}$.

(3.) The volume V_i of the i^{th} slab is approximately equal to the volume of the coin shape from part 2. We can approximate the volume of the cone as the sum of the approximate volumes of the different slabs:

$$V = \sum_{n=1}^{\infty} V_i \approx \sum_{n=1}^{\infty} \boxed{\frac{\pi r^2 (x_i^*)^2}{h^2}\,\Delta x}.$$

(4.) Use part 3 to express the volume of the cone as an integral.

(5.) Evaluate the integral from part 4 to find the volume of the cone.

$$V = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{\pi r^2 (x_i^*)^2}{h^2}\,\Delta x\right) = \boxed{\int_0^h \frac{\pi r^2 x^2}{h^2}\,dx} = \frac{\pi r^2 x^3}{3h^2}\Big|_{x=0}^{x=h} = \boxed{\frac{\pi r^2 h}{3}}.$$

Example: Find the volume of the solid obtained by revolving the region under one arch of the sine function around the x-axis.

We get one arch from $0 \leq x \leq \pi$. Using volumes by slicing, our volume will be

$$\int_0^\pi A(x)\, dx,$$

where $A(x)$ is the cross-sectional area at x. Since we got this solid by revolving our region around the x-axis, our cross-section will be a disc, with radius $\sin x$, so

$$A(x) = \pi \sin^2(x).$$

$$V = \int_0^\pi \pi \sin^2(x)\, dx = \frac{\pi^2}{2}.$$

We could have done this with the region under any graph $y = f(x)$:

Theorem (volumes by discs):

If a solid is generated by revolving the region under the curve $y = f(x)$ for $a \leq x \leq b$ around the x-axis, its volume is

$$\int_a^b \pi (f(x))^2 \, dx$$

Note: If you are revolving the region above $y = f(x)$ and below $y = g(x)$ around the x-axis, instead of a disc, the cross-section perpendicular to the x-axis will be a disc with a smaller disc removed from the center, the shape of the piece of hardware known as a washer. We compute the cross-sectional area as the area of the larger disc minus the area of the smaller disc that has been removed. You may see this referred to as "volumes by washers."

Example: Find the volume obtained by revolving the region bounded by the line $x = 1$ and the curve $x = y^2$ around the line $x = 1$.

Note: Be careful. We are not revolving around the x-axis this time.

Example: Find the volume that is generated by revolving the region under the curve $y = \sin(x)$ for $0 \leq x \leq \pi$ around the y-axis.

Divide the x-axis interval $[a, b] = [0, \pi]$ into n subintervals of length Δx. Vertical lines parallel to the y-axis at each division point divide our region into n-many strips. If we imagine revolving each of these strips around the y-axis, we divide our solid into a collection of thin cylindrical shells.

Approximate the volume of the i^{th} shell:

Choose x_i^* in the i^{th} subinterval. Approximate the i^{th} strip as a rectangle of height $f(x_i^*) = \sin(x_i^*)$ and width Δx. Revolving this strip around the y-axis gives a cylindrical shell of thickness Δx and outer surface area approximately that of a cylinder of height $f(x_i^*)$ and radius x_i^*:

$$V_i \approx (2\pi f(x_i))x_i \, \Delta x = (2\pi \sin(x_i^*)) \, x_i^* \, \Delta x.$$

Approximate the total volume as a sum:

$$V = \sum_{i=1}^{n} V_i \approx \sum_{i=1}^{n} (2\pi f(x_i^*))x_i^* \Delta x = \sum_{i=1}^{n} (2\pi \sin(x_i^*))x_i^* \, \Delta x.$$

Find the volume by taking a limit as the number of slices approaches infinity.

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2\pi f(x_i^*))x_i^* \, \Delta x = \int_a^b (2\pi f(x))x \, dx = \int_0^\pi (2\pi \sin(x))x \, dx =$$

$$2\pi(\sin(x) - x\cos(x))\Big|_{x=0}^{x=\pi} = 2\pi^2$$

Theorem (volumes by shells):

If all points of a solid are between a and b units from line ℓ, and the portion of the solid a distance x from ℓ is a cylinder of height $f(x)$, then the volume of the solid is

$$\int_a^b (2\pi f(x))x \, dx.$$

In particular, if the solid is generated by revolving region under the curve $y = f(x)$ for $a \leq x \leq b$ around the y-axis, its volume is

$$\int_a^b (2\pi f(x))x \, dx$$

Exercise: Find the volume of the solid obtained by revolving the region under the curve curve $y = \sqrt{x}$ for $0 \leq x \leq 4$ around the x-axis. Use two different methods:

1. Volumes by slicing;

2. Volumes by shells (noting that the roles of the x- and y-axes have been reversed from our example).

Suggestion: *Always* try to draw the solid whose volume you are finding. In a problem like this, start by sketching the region you are revolving around the axis.

Exercise: Use volumes by shells to find the volume of the region obtained by revolving the region under the line $y = \left(h - \frac{h}{r}x\right)$ for $0 \leq x \leq r$ around the y-axis.

Do you recognize this as a volume you already found by another method?

Exercise: Use volumes by slicing to find the volume of a pyramid whose base is a square of side 8, whose height is 5, and whose top point is directly above the center of the base.

Hint: Draw the x-axis from the top point of the pyramid through the center of the base, pointing downward, with 0 at the top point. Use similar figures to find the dimensions of cross-sections of the pyramid.

Exercise: Find the volume obtained by revolving the region under the line $y = 1$ and above the curve $y = x^2$ around the line $y = 1$. Use volumes by shells this time.

Solution to Challenge Problem from Last Time: Follow these steps to find a formula for the length of the curve $y = f(x)$ for $a \le x \le b$. We will assume that $f'(x)$ exists and is continuous over this interval.

Drawing pictures will be useful for this problem.

First solve this problem: A right triangle in the xy-plane has a horizontal leg of length Δx, a vertical leg of length Δy, and a hypotenuse of slope m. Find the length of the hypotenuse in terms of m and Δx.

You should be able to write your answer in the form (____________)Δx, where m appears in the expression inside the parentheses, but Δx does not.

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Solution: Let h be the length of the hypotenuse. By the Pythagorean theorem,

$$h = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Now, we use

$$\text{slope } = \frac{\Delta y}{\Delta x} = m$$

and solve for Δy:

$$\Delta y = m\Delta x.$$

This gives

$$h = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2 + (m\Delta x)^2} = \sqrt{(\Delta x)^2(m^2+1)} = \sqrt{m^2+1}\,\Delta x.$$

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Now approximate the length of the curve as follows:

Break the x-axis interval $a \le x \le b$ up into n-many small x-axis intervals of length Δx. This breaks the curve up into n-many small pieces, one lying above each small x-axis interval.

Choose a point x_i^* in each small x-axis interval.

Approximate the length of the i^{th} small piece of the curve by pretending it is a line segment with slope $f'(x_i^*)$ lying above the i^{th} small x-axis interval. If Δx is small enough, the slope won't change very much over this small interval, so this will be a good approximation. (When we zoom in enough on the graph of a differentiable function, it looks a lot like a straight line.)

Use your answer to the "first solve this problem" problem to find the approximation.

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Solution: The segment of curve lying above the i^{th} interval is approximately the hypotenuse of a right triangle with horizontal leg of length Δx and hypotenuse of slope $m = f'(x_i^*)$. Therefore, by our earlier answer, this segment of the curve has length approximately

$$\sqrt{m^2+1}\,\Delta x = \sqrt{(f'(x_i^*))^2+1}\,\Delta x.$$

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Approximate the length of the entire curve by adding up the approximate lengths of the small pieces.

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Solution:

$$\text{length } \approx \sum_{i=1}^{n} \sqrt{(f'(x_i^*))^2+1}\,\Delta x.$$

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Now take a limit as $n \to \infty$, and express that limit as an integral.

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Solution:

$$\text{length } = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(f'(x_i^*))^2+1}\,\Delta x = \int_a^b \sqrt{(f'(x))^2+1}\,dx.$$

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You can test your answer by applying it to the function $f(x) = \dfrac{4x}{3}$ and the interval $0 \le x \le 3$. (Use geometry to see what your answer should be.)

You can also find this formula in our textbook (Chapter 8, Section 1).