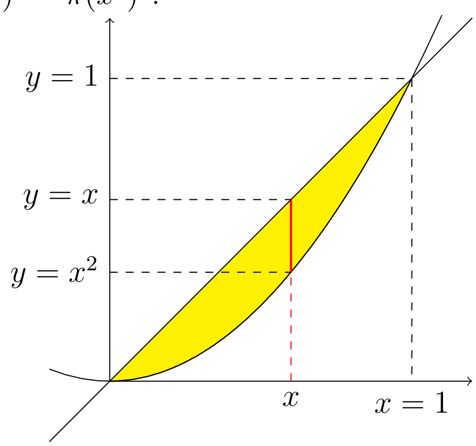
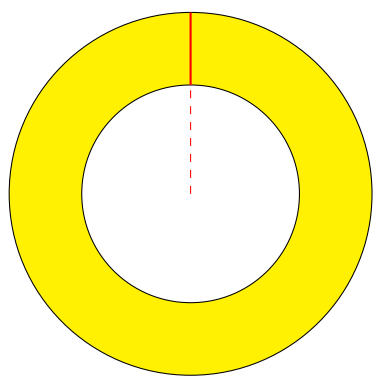
Math 8
Winter 2020
Section 1
January 23, 2020

First, some important points from the last class:

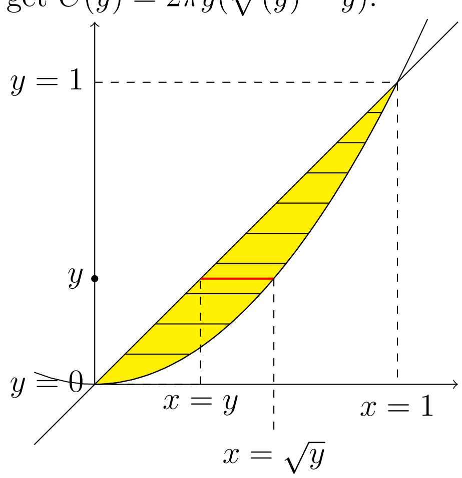
Two methods to compute volumes of revolution:

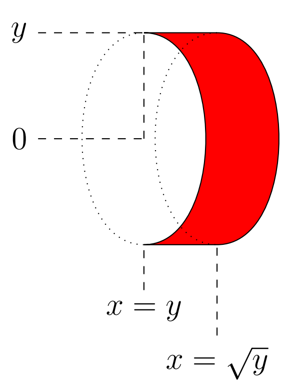
Volumes by discs or washers: Integrate the cross-sectional area along the axis around which you are revolving. In this picture, we revolve around the x-axis, and we get $\int_a^b A(x)\,dx$. The cross-section is a disc of radius x with a disc of radius x^2 removed. We have $A(x) = \pi(x)^2 - \pi(x^2)^2$.





Volumes by shells: Integrate the cylindrical-sectional area along an axis perpendicular to the one around which you are revolving. In this picture, we revolve around the x-axis, and we get $\int_c^d C(y)\,dy$. The cylindrical section is a cylinder of radius y and length $(\sqrt{(}y) - y)$. We get $C(y) = 2\pi y(\sqrt{(}y) - y)$.





Preliminary Homework:

1. Find the derivative of the function $f(x) = x\cos(x)$:

$$\frac{d}{dx}(x\cos(x)) = \cos(x) - x\sin(x)$$

2. Take the antiderivative with respect to x of both sides of the equation you found in part 1. Then evaluate every integral in the resulting equation except for $\int x\sin(x)\,dx$. (Don't forget the Fundamental Theorem of Calculus.)

$$\int \left(\frac{d}{dx}(x\cos(x))\right) dx = \int \cos(x)\,dx - \int x\sin(x)\,dx$$
$$(x\cos(x)) + C = \sin(x) - \int x\sin(x)\,dx$$

3. Use part 2 to find an antiderivative for $x\sin(x)$:

$$\int x\sin(x)\,dx = \sin(x) - x\cos(x) + C.$$

We can do this in general:

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$$
$$\int \frac{d}{dx}(f(x)g(x))\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$
$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}(f(x)g(x))\,dx - \int g(x)f'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Rewrite using $u = f(x) \quad v = g(x)$
$du = f'(x)\,dx \quad dv = g'(x)\,dx$:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

This method is called integration by parts. If we are computing a definite integral, we get

$$\boxed{\int_a^b u(x)\,\frac{dv(x)}{dx}\,dx = (u(x)v(x))\Big|_{x=a}^{x=b} - \int_a^b v(x)\,\frac{du(x)}{dx}\,dx}$$

Example: Find $\int x\sin(x)\,dx$:

$$\int \underbrace{\boxed{x}}_{u}\ \underbrace{\boxed{\sin(x)\,dx}}_{dv}$$

$$u = x \qquad dv = \sin(x)\,dx$$

$$du = dx \qquad v = -\cos(x)$$

$$\int x\sin(x)\,dx = \int u\,dv = uv - \int v\,du =$$

$$-x\cos(x) - \int -\cos(x)\,dx = -x\cos(x) + \int \cos(x)\,dx =$$

$$-x\cos(x) + \sin(x) + C$$

Example: Find $\int \ln x\,dx$:

$$\int \underbrace{\boxed{\ln x}}_{u}\ \underbrace{\boxed{dx}}_{dv}$$

$$u = \ln(x) \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = x$$

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du =$$

$$x\ln(x) - \int x \cdot \frac{1}{x}\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C$$

Strategy: Choose u to become simpler when differentiated, and dv to become no more complicated when integrated.

Example Find $\int e^x \cos x \, dx$.

Try integration by parts, with $u = e^x$ and $dv = \cos x \, dx$. This gives us $du = e^x \, dx$ and $v = \sin x$.

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

To find $\int e^x \sin x \, dx$, we try integration by parts again, with $u = e^x$ and $dv = \sin x \, dx$. This gives us $du = e^x \, dx$ and $v = -\cos x$.

$$\int e^x \sin x \, dx = e^x(-\cos x) - \int e^x(-\cos x) \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Substituting into our earlier equation, this gives

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx ==$$

$$e^x \sin x - \big(- e^x \cos x + \int e^x \cos x \, dx\big) =$$

$$e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

At first glance, we are back to finding our original integral $\int e^x \cos x \, dx$, which is not helpful. However, we can use this equation to solve for $\int e^x \cos x \, dx$, remembering there is an unwritten constant of integration:

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C;$$

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

Exercise: Use integration by parts to find $\int \tan^{-1}(x)\,dx$.

Hint: Look at the strategy we used to integrate $\ln(x)$, setting $dv = dx$. Recall that $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{x^2+1}$.

Exercise: Use volumes by shells to find the volume of the solid obtained by revolving the region below the curve $y = xe^x$, for $0 \leq x \leq 1$, around the y-axis. Note that we are revolving around the y-axis, so to use volumes by shells, we integrate along the x-axis.

Hint: To evaluate the integral, you may need to use integration by parts twice.

Note: We can't really use volumes by washers to find this volume. To find the cross-sectional area perpendicular to the y-axis, we would have to solve for x as a function of y in the equation $y = xe^x$.

Exercise: Use integration by parts to express $\int x^n\, e^x\, dx$ in terms of $\int x^{n-1}\, e^x\, dx$.

Use repeated applications of this formula to find $\int x^4\, e^x\, dx$.

Exercise: Suppose a piece of wire laid out along the portion of the x-axis $0 \leq x \leq \pi$ has mass density at point x given by the function

$$\rho(x) = e^x \sin(x).$$

Find the total mass of the piece of wire.