Math 8
Winter 2020
Section 1
January 24, 2020

First, some important points from the last class:

Theorem (integration by parts):

$$\boxed{\int u\,dv = uv - \int v\,du}$$

The strategy is to choose u to become simpler when differentiated, and v to become no more complicated when integrated.

Example:

$$\int \tan^{-1}(x)\,dx = \int u\,dv$$

$$u = \tan^{-1}(x) \qquad dv = dx$$

$$du = \frac{1}{x^2+1}\,dx \qquad v = x$$

$$\int u\,dv = uv - \int v\,du = x\tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx =$$

$$x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) + C.$$

Today: More Riemann sums and applications of integration.

Work:

If a force of magnitude F acts (in the direction of motion) on an object moving a distance d, the work done by that force on that object is the product of force and distance:

$$W = Fd.$$

If the force acts opposite to the direction of motion, the work it does is negative.

If the force is variable, or not all parts of the object move the same distance, we may use an integral to find work.

In SI units we measure distance in meters, force in newtons, and work in joules.

Newtons (N) and joules (J) are derived from the basic units of distance (meter, m), mass (kilogram, kg), and time (second, s).

$$\mathrm{N} = \frac{\mathrm{kg} \cdot \mathrm{m}}{\mathrm{s}^2} \qquad \mathrm{J} = \mathrm{N} \cdot \mathrm{m}.$$

$$\text{Work (in joules)} = \text{Force (in newtons)} \times \text{Distance (in meters)}.$$

Example: An object moves along the x-axis, from $x = a$ to $x = b$ (in meters). You may assume $a < b$.

When the object is at point x meters, it is acted on by a force of magnitude $F(x)$ newtons in the positive x-direction.

1. Approximate the work done on the object, by dividing its path up into small segments, approximating the work done as it moves along each segment, and taking a sum. Be sure to explain what you are doing. (If your answer involves a quantity such as Δx, say what Δx represents. Show how you approximated the work done along each segment.)

2. By taking an appropriate limit, express the work done on the object as a definite integral.

Note: Your answers above should include units.

This gives us a formula we can use: To find work, integrate force with respect to distance. If it makes sense (in the context of the problem) to treat an object as a single point moving along the x-axis from $x = a$ to $x = b$, acted on by a force that at point x has magnitude $F(x)$ in the x-direction (so the magnitude of the force depends only on the location x, not, for example, on time), then we can compute the work as

$$\boxed{Work = \int_a^b F(x)\,dx.}$$

When we cannot apply this formula:

When different parts of the object move different distances. (Example: pulling a rope to the top of a wall. We might be able to manipulate this one so we can apply the formula, but it does not naturally fit.)

When the force on the object depends on time, not just location. (Example: Launching a rocket that loses fuel, and therefore mass, as time passes. We might be able to use this formula with this problem if we could figure out how much fuel was lost when it reached height x, so we could express the mass, and therefore the force, as a function of x.)

In such situations, you can start from basics, approximating the work by a Riemann sum and then taking a limit to get an integral.

Example: The magnitude of the force of earth's gravity acting on a nearby object is $F = \dfrac{Cm}{r^2}$, where C is a constant (the product of the earth's mass and the universal gravitational constant), m is the mass of the object, and r is the distance from the object to the center of the earth.

How much work must be done against the force of gravity to launch a satellite of mass m from the surface of the earth (radius a) to height h above the earth's surface?

(Physics note: The force of gravity is not the only force acting opposite to the satellite's motion. There is also air resistance. This means the total work that must be done against the opposing forces is greater than just the work that must be done against gravity.

This is often the case with calculus work problems; we compute the work done by, or against, one particular force, which is not the only force acting. You can learn how to account for air resistance in a differential equations class.)

(Another physics note: Notice that the limit as $h \to \infty$ here is finite. Since work more or less equals energy, this means that imparting a finite amount of energy to the satellite will enable it to move as far away from the earth as you like. If you simply launch the satellite up from the earth's surface (rather than carrying it up via some propulsion mechanism), the amount of energy depends on its initial velocity. "Escape velocity" is the initial velocity sufficient to guarantee it will not fall back to the earth's surface.)

(Physics meta-note: The physics notes may not be technically complete, or completely precise. The principles involved should be correct, however, even though the subtleties may be glossed over.)

Probability Density Functions

If you roll a (fair) 6-sided die, there are 6 equally likely outcomes, so the probability of rolling a 1 is $\frac{1}{6}$ and the probability of rolling a 3 is $\frac{1}{6}$.

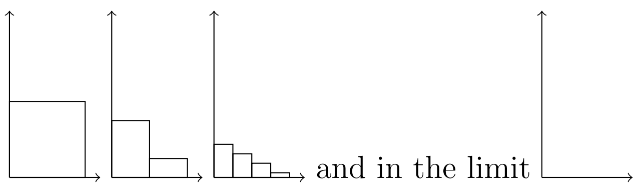
If you roll two dice, there are 11 possible outcomes for the sum of the numbers (from 2 to 12), but they are not equally likely. The probability of rolling a 2 is $\frac{1}{36}$, and the probability of rolling a 7 is $\frac{1}{6}$.

Now suppose you fly a drone over the green, release a small rubber ball from 20 feet above the center of the green, and measure the distance between the precise center of the green and the spot the ball comes to rest. There are infinitely many possible results, and they are not equally likely. The probability of any one precise number, of those infinitely many possibilities, would have to be zero. How can we talk about probability in this case?

First, we can talk about the probability of a result in a certain range, of being within one foot of the center, of being between one and two feet, and so forth. We can slice things more finely, by considering one inch instead of one foot intervals. It sounds like we could take a limit, which almost works. The problem is that as the length of the interval approaches zero, so does the probability of being in that interval.

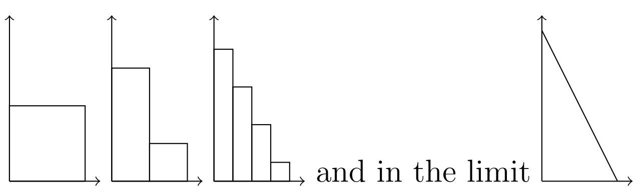
(Note: In probability theory an action with a random result, like rolling a die and seeing what number comes up, is often called an experiment.)

Here are some hypothetical graphs representing the probability of a certain event. For these pictures, it is an event whose result is between 0 and 1. For each picture, we have broken the interval $[0, 1]$ into subintervals, and graphed the function whose value above the i^{th} subinterval is the probability of being in that subinterval.

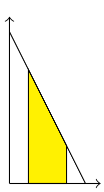


In these pictures, the height of the rectangle above the i^{th} subinterval is the probability of getting a result in that subinterval. The heights of the rectangles must add up to 1. So as the number of rectangles approaches infinity, the height of each individual rectangle approaches zero.

The solution is to instead make the *area* of the rectangle above the i^{th} subinterval equal the probability of getting a result in that subinterval. The areas of the rectangles must add up to 1. As the number of rectangles approaches infinity, the area covered remains equal to 1, but in the limit the top edge is a smooth curve.



The probability of a result in a given interval is the area below the curve on that interval.



Note: In order to have a probability density, the total area under the curve must equal 1.

Example: The probability density for the result of an experiment that yields positive values is given by $f(x) = e^{-x}$. What is the probability of getting a value less than 1?

We want the probability of a number between 0 and 1. Therefore we should integrate the probability density from 0 to 1. We get

$$\int_0^1 e^{-x}\,dx = -e^{-x}\Big|_{x=0}^{x=1} = -e^{-1} - (-e^0) = 1 - \frac{1}{e}.$$

Exercise: Near the surface of the earth, gravity acts on an object of mass m kilograms with a force of magnitude mg newtons directed downward. A leaky bucket is being lifted out of a well having a depth of 80 meters.

The bucket itself has mass 4 kilograms, and is initially (at the bottom of the well) filled with 80 kilograms of water. The water leaks out at a constant rate of .2 kilograms per second, while the bucket is lifted at a constant rate of 2 meters per second.

How much work must be done to lift the bucket out of the well?

Hint: Figure out the mass of the bucket when it has been lifted x meters. Then you will know how much force is acting on the bucket when it is x meters above the bottom of the well.

Note: We are ignoring the weight of the rope attached to the bucket.

Exercise: The expected value of a random experiment is the average value of the results if you perform the experiment many, many times. If the possible results of the experiment are n_1, n_2, ..., n_k, and the probability of getting value n_i is p_i, then the expected value is computed as

$$E = \sum_{i=1}^{k} n_i p_i.$$

For example, the possible results of rolling a die are 1, 2, 3, 4, 5, 6, and each one occurs with probability $\frac{1}{6}$, so the expected value is

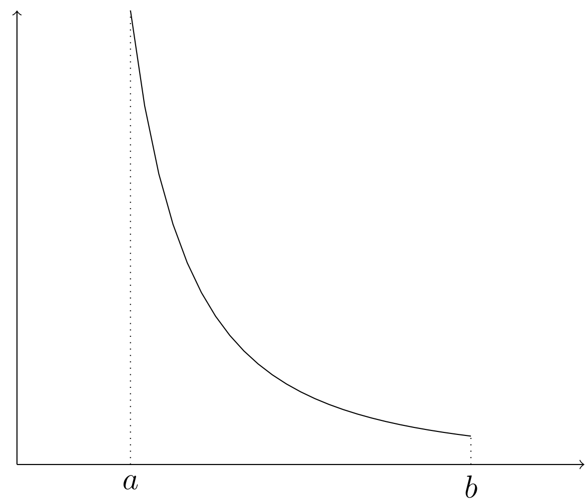
$$1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = 3.5.$$

(Notice that in this case, the expected value is not a value you ever expect from one roll of the die. It is what you expect as an average when you roll the die many times.)

How would we compute the expected value of an experiment with a probability density? We will assume the domain of the density is an interval $[a, b]$.

We will do the following: Divide the interval into many small intervals of length Δx. Choose x_i^* in interval number i. Pretend that the only possible values of the experiment are $x_1^*, x_2^*, \ldots, x_n^*$. Pretend the probability of getting x_i^* is the probability of getting any result in interval number i, which is the area under the curve above interval i, which is approximately the area of a rectangle above interval number i having height $f(x_i^*)$.

On the next page is a picture of a probability density. Add to this picture to illustrate the division into small intervals, and approximation of the probability of getting result x_i^*.



Write down a sum that approximates the expected value of an experiment with this probability density.

By taking an appropriate limit, find a definite integral that gives this expected value. This is called the *mean* of the probability density.

Suppose an experiment has a probability density given by $f(x) = \cos x$ for $0 \leq x \leq \frac{\pi}{2}$. Find the expected value for this experiment.

Exercise: The force exerted by a spring stretched x units beyond its natural length is kx, where k is a constant (called the spring constant) depending on the composition of the spring. (This is Hooke's Law, which holds as long as x is not too large.)

For example, if a spring has spring constant 200 newtons per meter, and its natural length is 5 meters, when it has been stretched to a length of 6 meters (1 meter beyond its natural length) it exerts a force of 200 newtons, and so 200 newtons of force must be applied to stretch it any further. Once it has been stretched to a length of 6.5 meters (1.5 meters beyond its natural length) it exerts a force of 300 newtons.

If the spring constant associated with a particular spring is 300 (in units of newtons per meter), how much work must be done to stretch the spring from its natural length of 20 centimeters to a length of 25 centimeters?

Suggestion: Rewrite the problem using meters instead of centimeters.
1 centimeter $= .01$ meter.

Challenge Problem: In a fantasy novel, a magician extends a very long golden filament (of length ℓ) from the earth's surface straight into the sky. One end of the filament is just touching the ground, and the other is a distance ℓ above the earth's surface, when the spell is disrupted by a passing dragon and the gold drops back to the surface of the earth.

The magnitude of the force of earth's gravity acting on a nearby object is $F = \dfrac{Cm}{r^2}$, where C is a constant (the product of the earth's mass and the universal gravitational constant), m is the mass of the object, and r is the distance from the object to the center of the earth. The radius of the earth is a. The linear mass density of the filament is a constant ρ (so a piece of filament of length w has mass ρw).

Assume the filament is long enough so the force of gravity on the far end is significantly different from the force on the near end. How much work is done by the force of gravity on the falling filament?

Suggestion: Break the filament into pieces of length Δx, and use the method of the satellite problem to find the work done on each falling piece. Then add them up and take a limit.