Math 8
Winter 2020
Section 1
January 24, 2020

The leaky bucket problem:

Near the surface of the earth, gravity acts on an object of mass m kilograms with a force of magnitude mg newtons directed downward. A leaky bucket is being lifted out of a well having a depth of 80 meters.

The bucket itself has mass 4 kilograms, and is initially (at the bottom of the well) filled with 80 kilograms of water. The water leaks out at a constant rate of .2 kilograms per second, while the bucket is lifted at a constant rate of 2 meters per second.

How much work must be done to lift the bucket out of the well?

Hint: Figure out the mass of the bucket when it has been lifted x meters. Then you will know how much force is acting on the bucket when it is x meters above the bottom of the well.

Note: We are ignoring the weight of the rope attached to the bucket.

Solution 1: The bucket originally masses 84 kilograms. It is being lifted at a rate of 2 meters per second, and water is leaking out at .2 kilograms per second. Therefore the water is leaking out at a rate of .2 kilograms per 2 meters lifted, or .1 kilogram per meter lifted. Therefore, when the bucket has been lifted x meters, its mass is $84 - .1x$ kilograms. The force of gravity acting on the bucket is its mass times the constant g.

Now we draw the x-axis, with units of meters, extending up from the bottom of the well, with $x = 0$ at the bottom and $x = 80$ at the top. The bucket is moving from $x = 0$ to $x = 80$, and when it is at point x it has been lifted x meters so the force of gravity on the bucket is $-g(84 - .1x)$ newtons. The minus sign is because the force acts downwards, and the object moves upwards in the positive x-direction. We do work against gravity with an upwards force of $F(x) = g(84 - .1x)$ newtons.

We have a problem where we have a single object moving along the x-axis from $x = 0$ to $x = 80$, and we have the force represented as a function of x, so we can integrate force with respect to distance to find work:

$$\int_0^{80} g(84 - .1x)\,dx = g(84x - .2x^2)\Big|_{x=0}^{x=80} = g(84(80) - .2(80)^2) \text{ joules}$$

Solution 2: The bucket is being lifted at a rate of 2 meters per second, so if t measures the time in seconds after it begins rising, it reaches the top of the well at time $t = 40$.

We break the time interval from $t = 0$ to $t = 40$ into n-many small intervals of length Δt seconds. If t_i^* is a time in the i^{th} interval, then at time t_i^* the bucket has been rising for t_i^*-many seconds, so $.2t_i^*$ kilograms of water have leaked out (because water leaks out at a rate of .2 kilograms per second), and its mass is $84 - .2t_i^*$ kilograms. Over that time interval of time Δt the bucket rises a distance of $2\Delta t$ meters (because it rises at a rate of 2 meters per second). Therefore the work done on the bucket over the i^{th} time interval is approximately $g(84 - .2t_i^*)(2\Delta t)$ joules.

We approximate the work as a Riemann sum

$$W \approx \sum_{i=1}^{n} g(84 - .2t_i^*)(2\Delta t)$$

and take a limit as $n \to \infty$ to get an integral

$$W = \int_0^{40} g(84 - .2t)2\,dt \text{ joules.}$$

Relating These Solutions: Since the bucket begins at distance $x = 0$ at time $t = 0$ and rises at a rate of 2 meters per second to distance $x = 80$ at time $t = 40$, we can relate x to t:

$$x = 2t \qquad t = \frac{x}{2}.$$

If you take the integral from solution 2, and use $x = 2t$, $dx = 2dt$, to make a direct substitution, you get the integral from solution 1. On the other hand, if you take the integral from solution 1, and use $t = \frac{x}{2}$, $dt = \frac{1}{2}\,dx$ to make a direct substitution, you get the integral from solution 1.

You can do something like this in general. If an object moves from point $x = a$ to point $x = b$ between times $t = c$ and $t = d$ at velocity $\frac{dx}{dt}$, during which time it is acted on by a variable force F, you can compute the work done by F as

$$\int_a^b F\,dx$$

or as

$$\int_c^d F\frac{dx}{dt}\,dt.$$

Which you choose will depend on whether you can most easily express F as a function of time or of distance.

You do not have to remember this second formula (for computing work by integrating over time). You can always arrive at it the way we did in the second solution to the bucket problem, by approximating the work by a Riemann sum and thus arriving at an integral.