Math 8
Winter 2020
Section 1
January 24, 2020

Solution to an exercise:

Exercise: The expected value of a random experiment is the average value of the results if you perform the experiment many, many times. If the possible results of the experiment are n_1, n_2, $\ldots$, n_k, and the probability of getting value n_i is p_i, then the expected value is computed as

$$E = \sum_{i=1}^{k} n_i p_i.$$

For example, the possible results of rolling a die are 1, 2, 3, 4, 5, 6, and each one occurs with probability $\frac{1}{6}$, so the expected value is

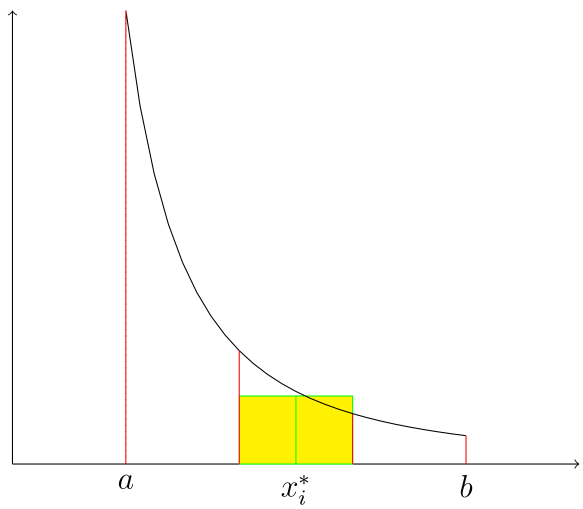
$$1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = 3.5.$$

(Notice that in this case, the expected value is not a value you ever expect from one roll of the die. It is what you expect as an average when you roll the die many times.)

How would we compute the expected value of an experiment with a probability density? We will assume the domain of the density is an interval $[a, b]$.

We will do the following: Divide the interval into many small intervals of length Δx. Choose x_i^* in interval number i. Pretend that the only possible values of the experiment are $x_1^*, x_2^*, \ldots, x_n^*$. Pretend the probability of getting x_i^* is the probability of getting any result in interval number i, which is the area under the curve above interval i, which is approximately the area of a rectangle above interval number i having height $f(x_i^*)$.

On the next page is a picture of a probability density. Add to this picture to illustrate the division into small intervals, and approximation of the probability of getting result x_i^*.



Solution: The additions are in color. The area of the yellow rectangle represents the approximate pretend probability of getting result x_i^* (actually, the approximate probability of getting a result in interval i).

The width of this rectangle is Δx, and its height is $f(x_i^*)$. The (pretend) probability of result x_i^* is approximately $f(x_i^*)\,\Delta x$.

Write down a sum that approximates the expected value of an experiment with this probability density.

Solution: For each i take the value x_i* times the (pretend) probability of result x_i^*, and sum the results.

$$\sum_{i=1}^{n} x_i^* f(x_i^*)\,\Delta x$$

By taking an appropriate limit, find a definite integral that gives this expected value. This is called the *mean* of the probability density.

Solution:

$$\int_a^b x f(x)\,dx.$$

Suppose an experiment has a probability density given by $f(x) = \cos x$ for $0 \leq x \leq \dfrac{\pi}{2}$. Find the expected value for this experiment.

Solution:

$$\int_0^{\frac{\pi}{2}} x\cos x\,dx = (x\sin x + \cos x)\Big|_{x=0}^{x=\frac{\pi}{2}} = \frac{\pi}{2} - 1.$$