Math 8
Winter 2020
Section 1
January 10, 2020

Informal Limit Proof

Example: Show

$$\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)=\infty.$$

Solution: Here the n^{th} term of our sequence is $a_n=\left(\frac{n^n}{n!}\right)$. (Notice that this number is always positive.) The ratio of successive terms is

$$\frac{a_{n+1}}{a_n}=\frac{\left(\frac{(n+1)^{(n+1)}}{(n+1)!}\right)}{\left(\frac{n^n}{n!}\right)}=\frac{n!}{(n+1)!}\frac{(n+1)^{n+1}}{n^n}=\frac{(n+1)^n}{n^n}=$$

$$\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n.$$

That is, multiply the term a_n by the number $\left(1+\frac{1}{n}\right)^n$ to get the next term a_{n+1}.

You may remember that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\approx 2.7$. This means that for all large enough n, the number $\left(1+\frac{1}{n}\right)^n$ is greater than 2, so $a_{n+1}>2a_n$. If we start with a positive number and double it over and over, it becomes larger and larger, with limit infinity. Therefore,

$$\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)=\infty.$$

Perhaps you don't remember that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$.

Example: Find $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Solution:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to\infty}e^{\ln\left(\left(1+\frac{1}{n}\right)^n\right)}=e^{\lim_{n\to\infty}\ln\left(\left(1+\frac{1}{n}\right)^n\right)}$$

We need to find $\lim_{n\to\infty} \ln\left(\left(1+\frac{1}{n}\right)^n\right)$.

$$\ln\left(\left(1+\frac{1}{n}\right)^n\right) = n\ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}.$$

We use l'Hôpital's Rule:

$$\lim_{n\to\infty} \ln\left(\left(1+\frac{1}{n}\right)^n\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{1}{n}\right)}\left(\frac{-1}{n^2}\right)}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{1}{n}\right)}}{1} = 1.$$

Therefore

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(\left(1+\frac{1}{n}\right)^n\right)} = e^{\lim_{n\to\infty}\ln\left(\left(1+\frac{1}{n}\right)^n\right)} = e^1 = e.$$