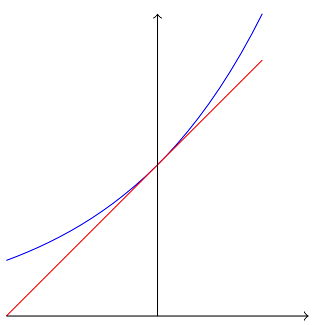
Math 8
Winter 2020
Taylor Polynomials and Taylor Series Day 1

You may recall the tangent line approximation from your study of derivatives. The tangent line approximation to the function $f(x)$ near the point $x = a$ is a function $T(x)$ whose graph is the line tangent to the graph of f at the point $(x, y) = (a, f(a))$. Another way to say this is that the function T has the same value and derivative as the function x at the point $x = a$. The formula for the tangent line approximation is

$$T(x) = f(a) + f'(a)(x - a).$$

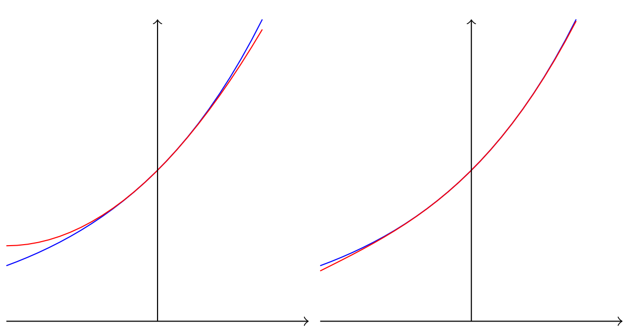
You can check that $T(a) = f(a)$ and $T'(a) = f'(a)$.

Below is a sketch of the graph of the function $f(x) = e^x$, in blue, and the graph of its tangent line approximation near $x = 0$, in red. You can see the tangent line approximation is fairly close to the function for x near 0, but less close when x is farther from 0.



We can think of the tangent line approximation as a polynomial of degree 1, chosen to have the same value and derivative as the function at the point $x = a$. This suggests a way to get a better approximation: Use a degree 2 polynomial that also has the same second derivative, or a degree 3 polynomial that also has the same second and third derivatives.

The two pictures below show these second- and third-degree approximations to $f(x) = e^x$ near $x = 0$.



Clearly we need not stop at degree 3. For any n, we can find a polynomial $T_n(x)$ of degree n that has the same value and first n derivatives as the function $f(x)$ at the point a.

This polynomial is called the n^{th} *Taylor polynomial* for $f(x)$ at a (or centered at a), and it is defined by

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{2\cdot 3}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{2\cdot 3\cdot 4\cdots n}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The expression $k!$, pronounced "k factorial," is defined by $k! = 1\cdot 2\cdot 3\cdot 4\cdots k$, except that for $k = 0$ we define $0! = 1$. The expression $f^{(k)}$ denotes the k^{th} derivative of f; we say $f^{(0)} = f$.

A Taylor polynomial centered at 0 is also called a Maclaurin polynomial.

Example: Compute some Taylor polynomials for the function $f(x) = \sin(x)$ centered at 0.

First we need to know some derivatives of our function at 0:

$$f(x) = \sin(x) \qquad f(0) = 0$$

$$f'(x) = \cos(x) \qquad f'(0) = 1$$

$$f''(x) = -\sin(x) \qquad f''(0) = 0$$

$$f^{(3)}(x) = -\cos(x) \qquad f^{(3)}(0) = -1$$

$$f^{(4)}(x) = \sin(x) \qquad f^{(4)}(0) = 0$$

$$\vdots$$

$$T_0(x) = f(0) = 0$$

$$T_1(x) = f(0) + f'(0)(x-0) = x$$

$$T_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = x$$

$$T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 = x - \frac{x^3}{6}$$

We can see the pattern in the derivatives of f at 0 (they are $0, 1, 0, -1, 0, 1, 0, -1 \ldots$), so we can write down any Taylor polynomial we want:

$$T_{10}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}.$$

We can even write down a general formula:

$$T_n(x) = \sum_{k=0}^{n} \frac{c_k}{k!}x^k \text{ where } c_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ (-1)^{\frac{n-1}{2}} & \text{if } k \text{ is odd.} \end{cases}$$

If we graph $T_n(x)$ for larger and larger values of n, the graphs of $T_n(x)$ look more and more like the graph of $f(x)$. (You can find an online program that will draw these graphs at <https://www.geogebra.org/m/frkWCH4U>. Or search for "Taylor Polynomials Geogebra.")

However, if we do the same thing with the function $f(x) = \ln(x)$ centered at 1, the graphs of $T_n(x)$ look more and more like the graph of $f(x)$ on the interval $0 < x < 2$, but not for $x > 2$.

If we actually want to use $T_n(x)$ as an approximation for $f(x)$, we would like to know whether it is a good approximation. We'd also like to know just how good.

First, some notation. If $T_n(x)$ is the n^{th} Taylor polynomial for $f(x)$ at a, we let $R_n(x)$ denote the difference between the actual value $f(x)$ and the approximate value $T_n(x)$, called the n^{th} remainder:

$$R_n(x) = f(x) - T_n(x).$$

The error in the approximation is the absolute value of the remainder, $|R_n(x)|$. The approximation is good if the error is small. We hope that we can make the approximation as good as we want just by making n large enough. That is, we hope that

$$\lim_{n\to\infty} R_n(x) = 0,$$

or

$$\lim_{n\to\infty} T_n(x) = f(x).$$

Example:

$$\begin{aligned}
f(x) &= x^{-2} & f(1) &= 1 \\
f'(x) &= -2x^{-3} & f'(1) &= -2 \\
f''(x) &= 3 \cdot 2x^{-4} & f''(1) &= 2 \cdot 3 \\
f^{(3)}(x) &= -4 \cdot 3 \cdot 2x^{-5} & f^{(4)}(0) &= -2 \cdot 3 \cdot 4 \\
&\vdots \\
f^{(k)}(x) &= (-1)^k (k+1)! x^{-(k+2)} & f^{(k)}(1) &= (-1)^k (k+1)! \\
T_n(x) &= \sum_{k=0}^{n} \frac{(-1)^k (k+1)!}{k!} (x-1)^k = \sum_{k=0}^{n} (-1)^k (k+1)(x-1)^k \\
T_n(2) &= \sum_{k=0}^{n} (-1)^k (k+1)(2-1)^k = 1 - 2 + 3 - 4 + 5 - \cdots \pm (n+1).
\end{aligned}$$

Clearly in this case

$$\lim_{n\to\infty} T_n(2) \neq f(2).$$

Over the next few classes we will explore this question. In the meantime, here is another example:

Example: Let $f(x) = \dfrac{1}{1-x}$. We can[1] compute the n^{th} Maclaurin polynomial of f as

$$T_n(x) = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x} = f(x) - \frac{x^{n+1}}{1-x},$$

so the error in the approximation is $\dfrac{x^{n+1}}{1-x}$, which approaches 0 as $n \to \infty$ just in case $\lim_{n\to\infty} x^{n+1} = 0$.

For $x = 2$, we have $\lim_{n\to\infty} 2^{n+1} = +\infty \neq 0$, so the Maclaurin polynomials $T_n(2)$ do not approach $f(2)$ as $n \to \infty$.

On the other hand, for $x = \dfrac{1}{2}$, we have $\lim_{n\to\infty} \left(\dfrac{1}{2}\right)^{n+1} = \lim_{n\to\infty} \left(\dfrac{1}{2^{n+1}}\right) = 0$, so the Maclaurin polynomials $T_n\left(\dfrac{1}{2}\right)$ do approach $f\left(\dfrac{1}{2}\right)$ as $n \to \infty$.

You may be able to see that the Maclaurin polynomials $T_n(x)$ approach $f(x)$ as $n \to \infty$ just in case $|x| < 1$.

[1] You should verify this by finding the Maclaurin polynomials for f, and by checking that the formula $1 + x + x^2 + \cdots + x^n = \dfrac{1-x^{n+1}}{1-x}$ is correct.