# Math 8
# Winter 2020
# Taylor Polynomials and Taylor Series Day 2

**Definition:** An infinite *sequence* of numbers is a list of numbers $(a_0, a_1, a_2, \dots)$, which is written as $(a_n)_{n=0}^{\infty}$, or (in our textbook) as $\{a_n\}_{n=0}^{\infty}$.

Warning: $\{a_0, a_1, a_2, \dots\}$ almost always means a set, NOT a sequence. Our textbook is violating a standard convention in its choice of notation.

An infinite sequence may actually begin at any integer, so we could have

$$(a_3, a_4, a_5, \dots) = (a_n)_{n=3}^{\infty}\,.$$

A sequence may be described by giving enough entries to establish a pattern:

$$(1, -1, 1, -1, 1, -1, \dots)$$

or by giving a formula for $a_n$:

$$((-1)^n)_{n=0}^{\infty}\,, \quad \text{or} \quad a_n = (-1)^n.$$

We are particularly interested in sequences of Taylor polynomials

$$(T_0(x), T_1(x), T_2(x), T_3(x) \dots).$$

We hope to find that

$$\lim_{n\to\infty} T_n(x) = f(x).$$

Intuitively, this means that we can make the approximation $T_n(x)$ as close as we want to $f(x)$ by making $n$ large enough. This idea is captured by the following definition:

**Definition:** The sequence $(a_n)_{n=i}^{\infty}$ converges to the number $L$, or

$$\lim_{n\to\ \infty} a_n = L,$$

if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that[1]

$$n > N \implies |a_n - L| < \varepsilon.$$

Intuitively, $\varepsilon$ is how close to $L$ you want $a_n$ to be, and $N$ is how large you have to make $n$ to guarantee $a_n$ is that close. Generally, $N$ is a function of $\varepsilon$; the closer to the limit you want to get, the farther out in the sequence you need to go.

[1] $\mathbb{N}$ denotes the set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. The symbol $\in$ means "is an element of." The number 0 is sometimes included as a natural number and sometimes not, depending on the text.

**Example:** Use the definition of limit to show that $.99999\cdots = 1$.

First we must say what we mean by $.99999\cdots$. Mathematicians say

$$.999999\cdots = \lim_{n\to\infty} .\underbrace{99999}_{n \text{ places}}.$$

To show this limit is 1, according to the definition, we must do the following:

Given $\varepsilon > 0$, find $N$ so that whenever $n > N$ we have

$$|1 - .\underbrace{99999}_{n \text{ places}}| < \varepsilon.$$

Now

$$1 - .\underbrace{99999}_{n \text{ places}} = .\underbrace{00001}_{n \text{ places}} = \frac{1}{10^n}.$$

We want to make this less than $\varepsilon$. Using what we know about logarithms (in particular that they are increasing functions), we have

$$\frac{1}{10^n} < \varepsilon \iff \frac{1}{\varepsilon} < 10^n \iff \log_{10}\left(\frac{1}{\varepsilon}\right) < \log_{10}(10^n) = n.$$

Therefore, we can say:

Given $\varepsilon > 0$, choose any $N \geq \log_{10}\left(\frac{1}{\varepsilon}\right)$. Then for every $n > N$ we have

$$n > \log_{10}\left(\frac{1}{\varepsilon}\right);$$

$$10^n > \frac{1}{\varepsilon};$$

$$\varepsilon > 10^{-n} = 1 - .\underbrace{99999}_{n \text{ places}} = |1 - .\underbrace{99999}_{n \text{ places}}|;$$

and this is what we needed to show.

**Note:** You can find another example of a proof using the definition of limit, with some additional comments about how to write proofs like this, at the end of these notes.

**Definition:** A sequence that does not converge to any number is said to diverge.

Even if a sequence approaches $+\infty$ or $-\infty$, we still say it diverges.

**Definition:** The sequence $(a_n)_{n=i}^{\infty}$ diverges to $+\infty$, or approaches $+\infty$, or

$$\lim_{n\to\infty} a_n = +\infty,$$

if for every $M \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that

$$n > N \implies a_n > M.$$

You should be able to give a similar definition for

$$\lim_{n \to \infty} a_n = -\infty.$$

A divergent sequence may diverge to $+\infty$ or to $-\infty$, in which case we may say its limit is $+\infty$ or $-\infty$, or diverge without having any limit at all.

**Example:** You should convince yourself that

$$\lim_{n\to\infty} x^n = \begin{cases} +\infty & \text{if } x > 1; \\ 1 & \text{if } x = 1; \\ 0 & \text{if } |x| < 1; \\ \text{no limit} & \text{if } x \leq -1. \end{cases}$$

**Example:** If $c$ is constant, then

$$\lim_{n\to\infty} \left(\frac{c^n}{n!}\right) = \lim_{n\to\infty} \left(\frac{c}{1}\right)\left(\frac{c}{2}\right)\left(\frac{c}{3}\right)\cdots\left(\frac{c}{n}\right).$$

For $n > 2c$, the $(n+1)^{th}$ term of the sequence is less than half the $n^{th}$ term, and dividing something in half over and over again indefinitely gives a limit of 0. Therefore,

$$\lim_{n\to\infty} \left(\frac{c^n}{n!}\right) = 0.$$

On the other hand,

$$\lim_{n\to\infty} \left(\frac{n^n}{n!}\right) = \infty.$$

To see this, suppose for simplicity that $n$ is even, so $n = 2m$.

$$\left(\frac{(2m)^{(2m)}}{(2m)!}\right) = \left(\frac{2m}{1}\right)\left(\frac{2m}{2}\right)\left(\frac{2m}{3}\right)\cdots\left(\frac{2m}{m}\right)\left(\frac{2m}{m+1}\right)\cdots\left(\frac{2m}{2m}\right)$$

If we increase the denominator of a fraction we decrease its value, so this is greater than

$$\left(\frac{2m}{m}\right)\left(\frac{2m}{m}\right)\left(\frac{2m}{m}\right)\cdots\left(\frac{2m}{m}\right)\left(\frac{2m}{2m}\right)\cdots\left(\frac{2m}{2m}\right) = (2)^m(1)^m = 2^m = 2^{\frac{n}{2}},$$

which approaches infinity as $n$ approaches infinity.

**Example:** The $n^{th}$ Maclaurin polynomial (Taylor polynomial centered at $a = 0$) for $f(x) = \dfrac{1}{1-x}$ is

$$T_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n.$$

We can find an explicit formula for $T_n(x)$ (assuming $x \neq 1$) as:

$$\begin{aligned} T_n(x) &= \frac{1}{1-x}(1-x)T_n(x) \\ &= \frac{1}{1-x}((1-x)(1 + x + x^2 + \cdots + x^{n-1} + x^n)) \\ &= \frac{1}{1-x}((1)(1 + x + x^2 + \cdots + x^{n-1} + x^n) - (x)(1 + x + x^2 + \cdots + x^{n-1} + x^n)) \\ &= \frac{1}{1-x}((1 + x + x^2 + \cdots + x^{n-1} + x^n) - (x + x^2 + x^3 \cdots + x^n + x^{n+1})). \end{aligned}$$

We have a lot of cancellation of terms here ($x$ and $-x$, $x^2$ and $-x^2$, ..., $x^n$ and $-x^n$), leaving

$$T_n(x) = \frac{1}{1-x}((1) - (x^{n+1})) = \frac{1 - x^{n+1}}{1-x}.$$

By the previous example, if $|x| < 1$, then $\lim_{n\to\infty} x^{n+1} = 0$, so

$$|x| < 1 \implies \lim_{n\to\infty} T_n(x) = \frac{1}{1-x} = f(x).$$

This is just what we hoped for. On the other hand,

$$|x| \geq 1 \implies (T_n(x))_{n=0}^{\infty} \text{ diverges},$$

so $T_n(x)$ can be used to get good approximations to $f(x)$ only for $|x| < 1$.

For example,

$$T_n(2) = 1 + 2 + 4 + \cdots + 2^n = \frac{1 - 2^{n+1}}{1-2} = \frac{1 - 2^{n+1}}{-1} = 2^{n+1} - 1,$$

so $\lim_{n\to\infty} T_n(2) = +\infty$.

**Note:** For a limit of Taylor polynomials

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

we may write

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

A sum of infinitely many term is called an infinite series, and this one in particular is a Taylor series. We will look more at series later.

A point of terminology: In ordinary English, sequence and series can mean more or less the same thing, but in mathematical English they are different. A sequence is a list of things. A series is a list of things added together.

There is an extensive theory of sequences and series, most of which we will not see in Math 8. In this section, we state a few rules that make so much sense that we have already used some of them without saying so. They should remind you of limit rules from Math 3. That's because they're really the same rules in a different guise.

Generally the limits $A$ and $B$ in these rules are assumed to be numbers. The rules also apply to limits of $\infty$ and $-\infty$, as long as the expression you are evaluating is defined ($\infty + \infty = \infty$) rather than undefined ($\infty - \infty$ is undefined). Be warned that the quotient $\frac{\infty}{0}$ is undefined, *not* $\infty$. That is because if $a_n$ approaches $\infty$ and $b_n$ approaches 0 while oscillating between positive and negative values, then $\frac{a_n}{b_n}$ will also oscillate between positive and negative values, and therefore will not approach either $\infty$ or $-\infty$.

You are free to use these rules in any homework or exam problem (unless the instructions say otherwise, such as, "use the definition of limit.") You do not have to cite the rule by name, as long as you make clear what fact you are using.

Do not be intimidated by the length of this list. This is not a collection of facts to memorize. This is reassurance that your common sense conclusions about series and sequences are generally valid.

**Sequence Rules**

1. (constant sequence rule)

   If $(a_n)_{n=0}^{\infty}$ is the constant sequence with value $c$ (that is, $a_n = c$ for every $n$), then

$$\lim_{n\to\infty} a_n = c.$$

2. (constant multiple rule)

   If $c$ is a constant, then

$$\left(\lim_{n\to\infty} a_n = A\right) \implies \lim_{n\to\infty} (ca_n) = cA.$$

3. (addition and subtraction rules)

$$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = B\right) \implies \lim_{n\to\infty} (a_n \pm b_n) = A \pm B.$$

4. (multiplication rule)

$$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = B\right) \implies \lim_{n\to\infty} (a_n b_n) = AB.$$

5. (division rule)

   If $b_n \neq 0$ for all $n$ and $B \neq 0$, then

$$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = B\right) \implies \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}.$$

6. (continuous function rule)

   If $f$ is continuous at $A$, then

$$\left(\lim_{n\to\infty} a_n = A\right) \implies \lim_{n\to\infty}(f(a_n)) = f(A).$$

   For example, since $\lim_{n\to\infty} \frac{1}{n} = 0$, we know $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$.

7. (horizontal asymptote rule)

$$\left(\lim_{x\to\infty} f(x) = A\right) \implies \lim_{n\to\infty}(f(n)) = A.$$

8. (limit comparison)

   If $a_n \leq b_n$ for all $n$, then

$$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = B\right) \implies A \leq B.$$

9. (squeeze theorem)

   If $a_n \leq c_n \leq b_n$ for all $n$, then

$$\left(\lim_{n\to\infty} a_n = A \;\&\; \lim_{n\to\infty} b_n = A\right) \implies \lim_{n\to\infty}(c_n) = A.$$

10. (tail end rule)

    The sequences $(a_n)_{n=0}^{\infty}$ and $(a_n)_{n=k}^{\infty}$ have the same limit.

11. (decreasing differences rule)

    If $(a_n)_{n=0}^{\infty}$ converges, then $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$.

    The converse of this is false, as you can see from the sequence

$$1,\ 1\frac{1}{2},\ 2,\ 2\frac{1}{3},\ 2\frac{2}{3},\ 3,\ 3\frac{1}{4}, \ldots,$$

    which does not converge even though the differences of successive terms do approach zero.

    We usually use this rule to show divergence: If $\lim_{n\to\infty}(a_{n+1} - a_n) \neq 0$ then $(a_n)_{n=0}^{\infty}$ does not converge.

12. (subsequence rule)

    If $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$ (that means it is the same sequence but with some — possibly infinitely many — terms left out), then

    $$\left(\lim_{n\to\infty} a_n = A\right) \implies \lim_{n\to\infty}(b_n) = A.$$

    On the other hand, the original sequence may diverge even if the subsequence converges. For example,

    $$1, 1, 1, 1, 1, \ldots$$

    is a subsequence of

    $$1, 2, 1, 3, 1, 4, 1, 5, \ldots$$

13. (monotone sequence theorem)

    An increasing sequence must either converge to a limit or approach $+\infty$, and a decreasing sequence must either converge to a limit or approach $-\infty$.

The Definition of Limit: Example Proof

**Problem:** Use the definition of limit to show that

$$\lim_{n\to\infty} \frac{3n^2+n+2}{n^2+1} = 3.$$

**Solution:**
Given $\varepsilon > 0$, we must find $N$ so that whenever $n > N$ we have

$$\left| 3 - \frac{3n^2+n+2}{n^2+1} \right| < \varepsilon.$$

Now

$$\left| 3 - \frac{3n^2+n+2}{n^2+1} \right| = \left| \frac{(3n^2+3)-(3n^2+n+2)}{n^2+1} \right| = \left| \frac{-n+1}{n^2+1} \right| = \left| \frac{n-1}{n^2+1} \right|.$$

If $n$ is large enough $(n \geq 1)$, we have

$$\left| \frac{n-1}{n^2+1} \right| = \frac{n-1}{n^2+1} < \frac{n}{n^2+1}.$$

We want to make this less than $\varepsilon$. We have

$$\frac{n}{n^2+1} < \varepsilon \iff \frac{1}{\varepsilon} < \frac{n^2+1}{n} = n + \frac{1}{n}.$$

We note that

$$\frac{1}{\varepsilon} < n \implies \frac{1}{\varepsilon} < n + \frac{1}{n}.$$

Therefore, we can say:

Given $\varepsilon > 0$, choose any $N \geq \frac{1}{\varepsilon}$ such that also $N \geq 1$. Then we have

$$n > N \implies n > \frac{1}{\varepsilon} \implies n + \frac{1}{n} > \frac{1}{\varepsilon} \implies \frac{n^2+1}{n} > \frac{1}{\varepsilon} \implies \frac{n}{n^2+1} < \varepsilon$$

and this is what we needed to show.

**Notes:** A typical proof using the definition of the limit of a sequence might look very much like this one, including everything written in black, with the details depending on the particular problem.

The two lines beginning "if $n$ is large enough" represent an optional step in the reasoning: Assume $n$ is large enough $(n \geq 1)$ so you can do some useful algebraic manipulation (in this case, dropping the absolute value signs). Of course, if you make this assumption, then when you choose $N$ you must make it at least that large $(N \geq 1)$.

The two lines beginning "we note that" represent another optional step in the reasoning: Replace the thing you are trying to make large $(n + \frac{1}{n})$ by a slightly smaller, but simpler, thing $(n)$, and make that large. This can go the other way, too: Replace the thing you are trying to make small by a slightly larger, but simpler, thing, and make that small.