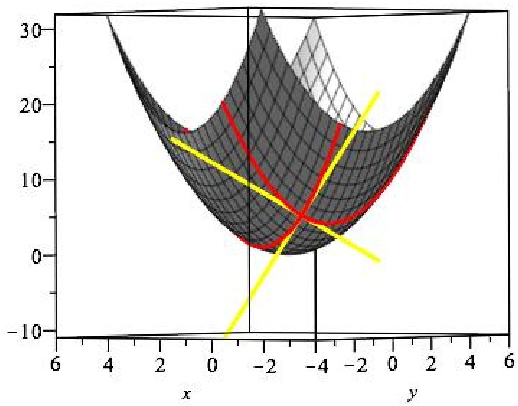



Figure 1: Finding a tangent plane to a graph

**Math 8**
**Winter 2020**
**Tangent Approximations and Differentiability**

We have seen how to find partial derivatives of functions from $\mathbb{R}^2$ to $\mathbb{R}$, and seen how to use them to find tangent planes to graphs.

Figure 1 shows the graph of the function $f(x, y) = x^2 + y^2$. The two red curves are the intersections of the graph with planes $x = x_0$ and $y = y_0$, and the yellow lines are tangent lines to the red curves, also lying in those planes. The slopes of the yellow lines (vertical rise over horizontal run, where the $z$-axis is vertical) are the partial derivatives of $f$ at the point $(x, y) = (x_0, y_0)$. The plane containing those yellow lines should be tangent to the graph of $f$.

The phrase *should be* is important here. So far, we have pretty much been assuming this works. There is a very good argument that if there is any plane tangent to this graph at this point, it must be the plane containing these yellow lines, because those lines are tangent to the graph. But how do we know there is any tangent plane at all?

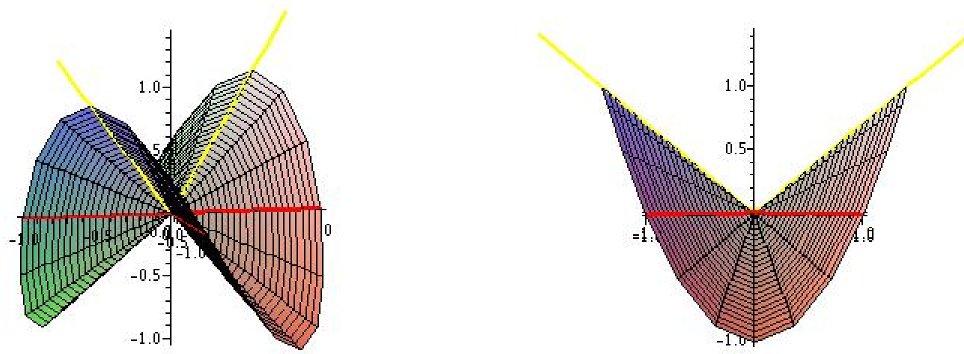



Figure 2: Graph of $f(x, y) = \dfrac{2xy}{\sqrt{x^2 + y^2}}$.
If $(x, y) = (r \sin \theta, r \cos \theta)$, then $f(x, y) = r \sin(2\theta)$.

It turns out that a function can have partial derivatives *without* its graph having a tangent plane.

Here is an example. Figure 2 shows two different pictures of a portion of the graph of the function

$$f(x, y) = \frac{2xy}{\sqrt{x^2 + y^2}}.$$

The $x$- and $y$-axes are drawn in red, and the intersection of the graph with the vertical plane $y = x$ is drawn in yellow.

The $x$- and $y$-axes lie on the graph of $f$. Therefore, if we were using the "plane containing two tangent lines" definition, we would conclude that the horizontal plane $z = 0$ is tangent to the graph of $f$ at the origin. However, as is most easily seen in the second picture, when we slice in the plane $x = y$, the graph of $f$ has a sharp point at the origin, like the absolute value function. Therefore, it cannot have a tangent plane.

**Definition:** A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *differentiable* at $(x, y) = (x_0, y_0)$ if the graph of $f$ has a non-vertical tangent plane at the point $(x_0, y_0, f(x_0, y_0))$.

**Fact:** If $f$ is differentiable at $(x_0, y_0)$, then both partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are defined.

This gives us a way to check that a function is *not* differentiable; just show one of the partial derivatives is undefined. There is also a way that sometimes works to check that a function *is* differentiable.

**Fact:** If the partial derivatives of $f$ are defined near $(x_0, y_0)$ (this means on some disc, even a very tiny one, with center $(x_0, y_0)$), and *continuous* at $(x_0, y_0)$, then $f$ is differentiable at $(x_0, y_0)$.

For example, the partial derivatives of any polynomial are also polynomials, and polynomials are defined and continuous everywhere. It follows that polynomials are differentiable everywhere.

Suppose the partial derivatives of $f$ are defined but not continuous at $(x_0, y_0)$. Then $f$ may or may not be differentiable at $(x_0, y_0)$. How would we check? Well, we said earlier that if the graph $f$ has any tangent plane, it must be the tangent plane defined using the partial derivatives of $f$.

**Fact:** If the partial derivatives of $f$ are defined at $(x_0, y_0)$, then $f$ is differentiable at $(x_0, y_0)$ just in case the plane with equation

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

is tangent to the graph of $f$ at the point $(x_0, y_0, f(x_0, y_0))$.

So, to check whether $f$ is differentiable at $(x_0, y_0)$, we should check whether this plane is actually tangent to the graph of $f$ at this point. Before we explore what this means, one more definition:

**Definition:** If $f$ is differentiable (and *only* then), the vector of its partial derivatives at $(a, b)$ is called its *total derivative*, or simply derivative, at $(x_0, y_0)$:

$f'(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$.

This lets us rewrite the definition of our candidate tangent plane in a familiar form:

$$z = f(x_0, y_0) + f'(x_0, y_0) \cdot \langle x - x_0, y - y_0\rangle.$$

So what does it mean for the plane

$$z = ax + by + d$$

to be tangent to the surface

$$z = f(x, y)$$

at some point $(x_0, y_0, z_0)$ that is on both the plane and the surface?

We answer this question for functions from $\mathbb{R}$ to $\mathbb{R}$ by defining the slope of the graph to be the limit of slopes of secant lines, and then saying a line is tangent to the graph if they have the same slope. More precisely, to find the slope of the graph of $f$ at $(x_0, f(x_0))$, we look at slopes of secant lines with one endpoint $(x_0, f(x_0))$ and the other endpoint $(x, f(x))$, and take the limit as $x \to x_0$:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

We will do something like this, but it's not quite this simple. We have already seen that the slopes of a surface in different directions are likely to be different. We can define the slope of a secant line with endpoints $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as (vertical) rise over (horizontal) run, where the run is the distance in the $xy$-plane between the points $(x_0, y_0)$ and $(x, y)$.

$$slope = \frac{f(x, y) - f(x_0, y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = \frac{f(x, y) - f(x_0, y_0)}{|(x, y) - (x_0, y_0)|}.$$

But then the limit

$$\lim_{(x,y) \to (x_0, y_0)} slope = \lim_{(x,y) \to (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0)}{|(x, y) - (x_0, y_0)|},$$

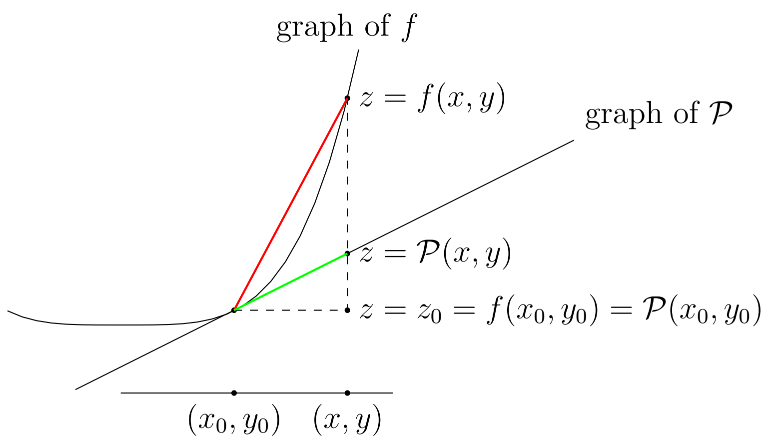
probably doesn't exist, even if the graph of $f$ *is* a plane.

Try it yourself, with the example $f(x, y) = x$ and $(x_0, y_0) = (0, 0)$. You can check that the limit

$$\lim_{(x,y) \to (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0)}{|(x, y) - (x_0, y_0)|} = \lim_{(x,y) \to (0,0)} \frac{f(x, y) - f(0, 0)}{|(x, y) - (0, 0)|} = \lim_{(x,y) \to (0,0)} \frac{x}{\sqrt{x^2 + y^2}}$$

does not exist.

However, we still want to use this idea. If a plane is tangent to the graph of $f$ at some point, then in the limit, secant lines of the graph of $f$ and line segments in the plane should have the same slope. We will say this by saying that, in the limit, the difference of the slopes is zero.




Suppose that the point $(x_0, y_0, z_0)$ lies on both the graph of $f$ and the plane

$$z = \mathcal{P}(x, y) = ax + by + d.$$

This means we must have

$$f(x_0, y_0) = \mathcal{P}(x_0, y_0) = z_0.$$

Taking a point $(x, y)$ close to $(x_0, y_0)$ in the domain of $f$ gives us a (red) secant line of the graph of $f$, with endpoints $(x_0, y_0, f(x_0, y_0))$, $(x, y, f(x, y))$ and slope

$$\frac{f((x, y) - f(x_0, y_0)}{|(x, y) - (x_0, y_0)|} = \frac{f((x, y) - z_0}{|(x, y) - (x_0, y_0)|},$$

and it also gives us a (green) line segment in the plane, with endpoints $(x_0, y_0, \mathcal{P}(x_0, y_0)$, $(x, y, \mathcal{P}(x, y))$ and slope

$$\frac{\mathcal{P}(x, y) - \mathcal{P}(x_0, y_0)}{|(x, y) - (x_0, y_0)|} = \frac{\mathcal{P}(x, y) - z_0}{|(x, y) - (x_0, y_0)|}.$$

The difference of these slopes is

$$\frac{f(x, y) - z_0}{|(x, y) - (x_0, y_0)|} - \frac{\mathcal{P}(x, y) - z_0}{|(x, y) - (x_0, y_0)|} = \frac{f(x, y) - \mathcal{P}((x, y)}{|(x, y) - (x_0, y_0)|}.$$

We say that the plane is tangent to the graph of $f$ at $(x_0, y_0, z_0)$ if this difference of slopes approaches 0 as $(x, y)$ approaches $(x_0, y_0)$:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (x_0,y_0)|} = 0.$$

**Definition:** If $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}$, then the graphs of $f$ and $\mathcal{P}$ are tangent at $(x_0, y_0, z_0)$ if and only if $\boxed{f(x_0, y_0) = \mathcal{P}(x_0, y_0) = z_0}$ and

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (x_0,y_0)|} = 0.}$$

**Definition:** The function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0, y_0)$ if its graph has a non-vertical tangent plane at the point $(x_0, y_0, f(x_0, y_0))$.

We use these same definitions for functions $f : \mathbb{R}^n \to \mathbb{R}$.

**Example:** Show that the graphs of the functions $f(x,y) = x^2 - y^2$ and $\mathcal{P}(x,y) = 2x - 4y + 3$ are tangent at the point $(1,2,-3)$.

What does this say about the derivative of $f$?

First we check that $(1,2,-3)$ is on both graphs:

$$f(1,2) = (1)^2 - (2)^2 = 1 - 4 = -3$$

$$\mathcal{P}(1,2) = 2(1) - 4(2) + 3 = 2 - 8 + 3 = -3.$$

Now we use the definition of tangent. Here our point is $(x_0, y_0) = (1,2)$, so we need to check that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (x_0,y_0)|} = \lim_{(x,y)\to(1,2)} \frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (1,2)|} = 0.$$

Evaluating the expression inside the limit:

$$\frac{f(x,y) - \mathcal{P}(x,y)}{|(x,y) - (1,2)|} = \frac{(x^2 - y^2) - (2x - 4y + 3)}{\sqrt{(x-1)^2 + (y-2)^2}} = \frac{x^2 - 2x + 1 - y^2 + 4y - 4}{\sqrt{(x-1)^2 + (y-2)^2}}$$

$$= \frac{(x-1)^2 - (y-2)^2}{\sqrt{(x-1)^2 + (y-2)^2}} = \frac{(x-1)^2}{\sqrt{(x-1)^2 + (y-2)^2}} - \frac{(y-2)^2}{\sqrt{(x-1)^2 + (y-2)^2}}$$

Looking at the first piece of this:

$$\frac{(x-1)^2}{\sqrt{(x-1)^2 + (y-2)^2}} = \frac{\sqrt{(x-1)^4}}{\sqrt{(x-1)^2 + (y-2)^2}} = \sqrt{\frac{(x-1)^4}{(x-1)^2 + (y-2)^2}} =$$

$$\sqrt{(x-1)^2 \left( \frac{(x-1)^2}{(x-1)^2 + (y-2)^2} \right)} \leq \sqrt{(x-1)^2} = |x-1|.$$

(The $\leq$ step is because the numerator of $\left( \frac{(x-1)^2}{(x-1)^2 + (y-2)^2} \right)$ is less than or equal to the denominator, so the fraction is at most 1.)

In the same way, we have

$$\frac{(y-2)^2}{\sqrt{(x-1)^2 + (y-2)^2}} \leq |y-2|,$$

and so

$$\left|\frac{f(x,y)-\mathcal{P}(x,y)}{|(x,y)-(1,2)|}\right| = \left|\frac{(x-1)^2}{\sqrt{(x-1)^2+(y-2)^2}} - \frac{(y-2)^2}{\sqrt{(x-1)^2+(y-2)^2}}\right|$$

$$\leq \left|\frac{(x-1)^2}{\sqrt{(x-1)^2+(y-2)^2}}\right| + \left|\frac{(y-2)^2}{\sqrt{(x-1)^2+(y-2)^2}}\right| \leq |x-1|+|y-2|.$$

Now

$$\lim_{(x,y)\to(1,2)} |x-1| = 0 \qquad \text{and} \quad \lim_{(x,y)\to(1,2)} |y-2| = 0,$$

and therefore

$$\lim_{(x,y)\to(1,2)} \left|\frac{f(x,y)-\mathcal{P}(x,y)}{|(x,y)-(1,2)|}\right| = 0,$$

and so,

$$\lim_{(x,y)\to(1,2)} \frac{f(x,y)-\mathcal{P}(x,y)}{|(x,y)-(1,2)|} = 0.$$

This is what we needed to show.

What does this say about the derivative of $f$?

Since the graph of $\mathcal{P}$ is a plane tangent to the graph of $f$ at the point where $(x,y)=(1,2)$, this says that $f$ is differentiable at $(1,2)$.

Furthermore, we can write

$$\mathcal{P}(x,y) = \langle 2,-4\rangle \cdot \langle x,y\rangle + 3,$$

and therefore

$$f'(1,2) = \langle 2,-4\rangle\,.$$

Pretty much all our derivatives and tangent approximations can be put in the same basket:

For $f : \mathbb{R} \to \mathbb{R}$, the tangent approximation at $x = x_0$ is given by
$\mathcal{T}(x) = f'(x_0)(x - x_0) + f(x_0)$

For $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, the tangent approximation at $t = t_0$ is given by
$\vec{\mathcal{T}}(t) = (t - t_0)\vec{r}\,'(t_0) + \vec{r}(t_0)$.

For $f : \mathbb{R}^n \to \mathbb{R}$ the tangent approximation at $(x, y, z \dots) = (x_0, y_0, z_0 \dots)$ is given by
$\mathcal{T}(x, y, z, \dots) = f'(x_0, y_0, z_0 \dots) \cdot \langle x - x_0, y - y_0, z - z_0, \dots \rangle + f(x_0, y_0, z_0 \dots)$.

**Note:** Our definition of differentiable is different from the textbook's, and that is because our definition of tangent is different. (The two definitions are equivalent, they are just phrased differently.)

Suppose we define $E(x,y) = f(x,y) - \mathcal{P}(x,y)$ to be the error in using $\mathcal{P}(x,y)$ as an approximation to $f(x,y)$ near $(x_0,y_0)$. Then our definition of tangent (assuming that $f(x_0,y_0) = \mathcal{P}(x_0,y_0)$) is that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{|(x,y)-(x_0,y_0)|} = 0.$$

The textbook's definition says we can write the error as the sum of two parts

$$E(x,y) = \varepsilon_1(x,y)(x-x_0) + \varepsilon_2(x,y)(y-y_0)$$

in such a way that

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1(x,y) = 0 \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} \varepsilon_2(x,y) = 0.$$

As we said, these two definitions are equivalent, and you may use either one.

In the example on page 7, we had $(x_0,y_0) = (1,2)$, and our error function $E(x,y) = f(x,y) - \mathcal{P}(x,y)$ was computed to be $(x-1)^2 - (y-2)^2$. We can write this as

$$E(x,y) = \underbrace{(x-1)}_{\varepsilon_1(x,y)}\underbrace{(x-1)}_{(x-x_0)} + \underbrace{(-(y-2))}_{\varepsilon_2(x,y)}\underbrace{(y-2)}_{(y-y_0)},$$

and check that

$$\lim_{(x,y)\to(1,2)} (x-1) = 0 \qquad \text{and} \qquad \lim_{(x,y)\to(1,2)} (-(y-2)) = 0.$$

This worked out nicely for this example. However, suppose we want to show that the graph of $\mathcal{P}(x,y) = x+y$ is tangent to the graph of $f(x,y) = \sin(x+y)$ at $(0,0)$. We can find

$$E(x,y) = f(x,y) - \mathcal{P}(x,y) = \sin(x+y) - x - y.$$

It's hard to see how we should rewrite this in the form $\varepsilon_1(x,y)(x-0) + \varepsilon_2(x,y)(y-0)$. It's also hard to see how to show

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)-x-y}{|(x,y)-(0,0)|} = 0,$$

but we do know what we are trying to do.