Math 8
Winter 2020
Taylor Polynomials and Taylor Series Day 3

Infinite Series

Our major interest in discussing limits of sequences is to find limits of Taylor polynomials,

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

We may write this limit instead as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Definition: An infinite series is a sum of infinitely many terms,

$$\sum_{k=0}^{\infty} a_k.$$

(A sequence is a list; a series is a sum.) The sum is defined as follows: The n^{th} partial sum is the sum of the first n terms,

$$S_n = \sum_{k=0}^{n-1} a_k,$$

and the sum of the series is the limit of the partial sums,

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} a_k.$$

If this limit exists and is a number, the series converges; if not, it diverges.

Definition: The Taylor series for $f(x)$ centered at a is the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This is sort of the degree infinity Taylor polynomial. Its n^{th} partial sum is the degree $n-1$ Taylor polynomial for $f(x)$ centered at a.

Remark: We hope that the sum of the Taylor series for $f(x)$ is equal to $f(x)$. In many nice cases, it is. We will learn of some important examples where this is true.

Geometric Series

Definition: The series $\sum_{n=0}^{\infty} a_n$ is a *geometric series* with ratio r if for every n we have $\frac{a_{n+1}}{a_n} = r$ (for the same number r); in other words, $a_{n+1} = a_n r$.

If $\sum_{n=0}^{\infty} a_n$ is a geometric series with ratio r, we can write

$$a_1 = a_0 r \qquad a_2 = a_1 r = (a_0 r) r = a_0 r^2 \qquad a_3 = a_2 r = (a_0 r^2) r = a_0 r^3 \qquad \cdots \qquad a_n = a_0 r^n.$$

Therefore, we can write the series as

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_0 r^n = a_0 \sum_{n=0}^{\infty} r^n.$$

But $\sum_{n=0}^{\infty} x^n$ is just the Taylor series centered at 0 for the function $f(x) = \frac{1}{1-x}$, and we have already seen that

$$\lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \begin{cases} \frac{1}{1-x} & \text{if } |x| < 1; \\ \text{divergent} & \text{if } |x| \geq 1. \end{cases}$$

This gives us the following proposition.

Proposition:
If $\sum_{n=0}^{\infty} a_n$ is a geometric series with ratio r and first term a_0, then

$$\sum_{n=0}^{\infty} a_n = \begin{cases} \frac{a_0}{1-r} & \text{if } |r| < 1; \\ \text{divergent} & \text{if } |r| \geq 1. \end{cases}$$

Example: Find the sum of the series

$$\sum_{k=0}^{\infty} \frac{2 \cdot 3^k}{5^{2k}}.$$

The k^{th} term of the series is $a_k = \dfrac{2 \cdot 3^k}{5^{2k}}$, and the ratio of successive terms is

$$\frac{a_{k+1}}{a_k} = \frac{\frac{2 \cdot 3^{k+1}}{5^{2(k+1)}}}{\frac{2 \cdot 3^k}{5^{2k}}} = \frac{2 \cdot 3^{k+1} 5^{2k}}{2 \cdot 3^k 5^{2(k+1)}} = \frac{3}{25}.$$

Because this is the same number for every k, this is a geometric series, with ratio $r = \dfrac{3}{25}$ and $|r| < 1$. The first term is $a_0 = \dfrac{2 \cdot 3^0}{5^0} = 2$, so the sum is

$$\sum_{k=0}^{\infty} \frac{2 \cdot 3^k}{5^{2k}} = \frac{a_0}{1-r} = \frac{2}{\frac{22}{25}} = \frac{25}{11}.$$

Some Rules for Series

There is an extensive theory of sequences and series, most of which we will not see in Math 8. In this section, we state a few rules that should make sense. This is not a collection of facts to memorize. This is reassurance that your common sense conclusions about series and sequences are generally valid.

Generally the limits A and B in these rules are assumed to be numbers. The rules also apply to limits of ∞ and $-\infty$, as long as the expression you are evaluating is defined ($\infty + \infty = \infty$) rather than undefined ($\infty - \infty$ is undefined). Be warned that the quotient $\dfrac{\infty}{0}$ is undefined, *not* ∞. That is because if a_n approaches ∞ and b_n approaches 0 while oscillating between positive and negative values, then $\dfrac{a_n}{b_n}$ will also oscillate between positive and negative values, and therefore will not approach ∞.

You are free to use these rules in any homework or exam problem (unless the instructions say otherwise, such as, "use the definition of limit.") You do not have to cite the rule by name, as long as you make clear what fact you are using.

1. (constant multiple rule)

   If c is a constant, then

$$\left(\sum_{n=0}^{\infty} a_n = A \right) \implies \sum_{n=0}^{\infty} (c a_n) = cA.$$

2. (addition and subtraction rules)

$$\left(\sum_{n=0}^{\infty} a_n = A \ \& \ \sum_{n=0}^{\infty} b_n = B \right) \implies \sum_{n=0}^{\infty} (a_n \pm b_n) = A \pm B.$$

3. (tail end rule)
$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \sum_{n=k}^{\infty} a_n \text{ converges.}$$
In fact,
$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

4. (comparison rule)
If $a_n \leq b_n$ for all n, then
$$\left(\sum_{n=0}^{\infty} a_n = A \ \& \ \sum_{n=0}^{\infty} b_n = B \right) \implies A \leq B.$$

5. (decreasing terms rule)
If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{n\to\infty} (a_k) = 0$.
The converse of this is false, as you can see from the series
$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots ,$$
which does not converge even though the individual terms do approach zero.

6. (nonnegative series rule)
If $a_n \geq 0$ for all n, then $\sum_{n=0}^{\infty} a_n$ either converges to a finite sum or approaches $+\infty$.

These rules basically follow from applying sequence rules to the sequences of partial sums. For example, the nonnegative series rule follows from the monotone sequence theorem, since if $a_n \geq 0$ for all n, then the sequence of partial sums is an increasing sequence, which must either converge to a number or diverge to $+\infty$.

Convergence

You may notice that we have listed many more sequence rules than series rules. This is because there are many more elementary methods for finding the limit of a sequence than for finding the sum of a series.

Sometimes, even if we cannot find the sum of a series, we can determine whether the series converges or not. There are a number of different convergence tests for series. We will see a few of them now.

Be warned, the next proposition applies *only* to nonnegative series, which are series with no negative terms.

Proposition (the comparison test): Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are non-negative series. If $0 \le b_n \le a_n$ for all n, then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \implies \sum_{n=0}^{\infty} b_n \text{ converges.}$$

This proposition follows from the fact that a series with nonnegative terms must either converge or approach infinity. If the larger series does not approach infinity, the smaller one cannot do so either, so it must converge.

The following definition turns out to be useful.

Definition: The series $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.

The reason it is useful is the following proposition.

Proposition: If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then it is convergent.

This proposition follows from things we have already noted. By the sum rule, we can break up a series into the sum of its positive terms and the sum of its negative terms, and show each of those series converges by comparison to the sum of the absolute values of the terms. (There are more details in the last section.)

Be warned that the converse of this proposition is false. There are some series that are convergent but not absolutely convergent. The alternating harmonic series,

$$\sum_{n=1}^{\infty} \left((-1)^{n+1} \left(\frac{1}{n} \right) \right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots ,$$

is one important example.

We can use this proposition to prove $\sum_{n=0}^{\infty} a_n$ converges by proving $\sum_{n=0}^{\infty} |a_n|$ converges. This is often easier, because we have tests such as the comparison test that apply only to nonnegative series.

Alternating Series

Here is a convergence test that applies to series that are not non-negative. It applies to alternating series, series whose terms alternate between positive and negative.

For an example of an alternating series, remember that the Maclaurin polynomials (Taylor polynomials with center 0) for $\cos(x)$ have the form

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

so we can write the Maclaurin series for $\cos(x)$ as

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}.$$

Whatever value we use for x, the terms of this series alternate between positive and negative.

Proposition (the alternating series test): If a series $\sum_{k=0}^{\infty} a_k$ satisfies the following three conditions, then it converges:
(1.) The terms a_n alternate between positive and negative.
(2.) The terms a_n are decreasing in absolute value, that is, $|a_{n+1}| \leq |a_n|$ for all n.
(3.) The terms a_n are approaching zero, $\lim_{n\to\infty} a_n = 0$.

There is a nice proof, with picture, at the beginning of section 11.5 in the textbook.

Example: Show that the Maclaurin series for $\cos(x)$ converges for $x = 1$.

This series, we see by substituting $x = 1$ into the Maclaurin series we had above, is

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}.$$

It is easy to see that all three conditions of the alternating series test are satisfied.

Example: Show that the Maclaurin series for $\cos(x)$ converges for $x = 10$.

This is a harder problem, because this series doesn't quite satisfy all the conditions of the alternating series test. The series is

$$\sum_{k=0}^{\infty}(-1)^k \frac{10^{2k}}{(2k)!}.$$

The terms do alternate between positive and negative, and they do approach zero, since we have already seen that if c is any constant then $\lim_{n\to\infty} \frac{c^n}{n!} = 0$. However, the first few terms are

$$1, -\frac{100}{2}, \frac{10,000}{24}, \ldots,$$

which are clearly not getting smaller in absolute value.

From the tail end test, it is enough to show that the tail end

$$\sum_{k=5}^{\infty}(-1)^k \frac{10^{2k}}{(2k)!}$$

converges. For this series, we have $a_k = (-1)^k \dfrac{10^{2k}}{(2k)!}$, and we can write

$$|a_{k+1}| = \frac{10^{2k+2}}{(2k+2)!} = \left(\frac{10 \cdot 10 \cdot 10^{2k}}{(2k+2)(2k+1)(2k)!}\right) = \left(\frac{10}{2k+2}\right)\left(\frac{10}{2k+1}\right) a_k,$$

which is less than a_k, because $k \geq 5$ and so $\dfrac{10}{2k+2} < 1$ and $\dfrac{10}{2k+1} < 1$.

Sums of Taylor Series

There are at least three methods to determine what a Taylor series converges to.

Special Examples

We might recognize a series we know about, or a series we can analyze in some clever way. For example, the Maclaurin series for $f(x) = \dfrac{1}{x^2+1}$ is

$$1 - x^2 + x^4 - x^6 + \cdots .$$

This is a geometric series with first term 1 and ratio $r = -x^2$, so we know it converges to $\dfrac{1}{1-(-x^2)} = \dfrac{1}{x^2+1} = f(x)$ for $|-x^2| < 1$ (that is, for $|x| < 1$) and diverges for $|x| \geq 1$. For $x = 1$ or $x = -1$ we can substitute 1 or -1 for x, and see that the series we get $(1 - 1 + 1 - 1 + 1 \cdots)$ diverges.

Taylor's Inequality

Taylor's Inequality is a method of finding a bound on how big the error in using a Taylor polynomial to approximate a function,

$$|R_n(x)| = |f(x) - T_n(x)|$$

could possibly be. We can sometimes use this to prove that the error must approach zero as $n \to \infty$.

We are not going to cover Taylor's Inequality in Math 8. (You can read about it in Sections 11.10 and 11.11 of the textbook.) However, we will note some important things that can be proven from Taylor's Inequality. You should remember this fact:

Proposition: If a is any number, then, for the functions $f(x) = \sin(x)$, $f(x) = \cos(x)$, and $f(x) = e^x$, the Taylor series for f centered at a converges to $f(x)$ for all values of x.

Example: The Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

By the proposition, for every x we have

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

In particular, taking $x = 1$, we get

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e^1 = e$$

This would give us a method for approximating e to as many decimal places as we like, **IF** we had a way to determine how close the partial sums $\sum_{k=0}^{n} \frac{x^k}{k!}$ are to the actual sum. We will talk about how we might determine this next time.

Integrals and Derivatives

We will see this method later.

Proof that absolutely convergent series are convergent:

Let $\sum_{n=0}^{\infty} a_n$ be any series. Some terms may be positive and some negative.

Suppose that $\sum_{n=0}^{\infty} |a_n|$ converges. We want to show the original series $\sum_{n=0}^{\infty} a_n$ also converges.

We define two new series, one including all the positive terms of our original series, and the other including all the (absolute values of) negative terms:

$$b_n = \begin{cases} a_n & \text{if } a_n \geq 0; \\ 0 & \text{if } a_n < 0. \end{cases} \qquad c_n = \begin{cases} 0 & \text{if } a_n \geq 0; \\ -a_n & \text{if } a_n < 0. \end{cases}$$

It is not hard to check that for every n we have:

$$(1.)\ 0 \leq b_n \leq |a_n| \qquad (2.)\ 0 \leq c_n \leq |a_n| \qquad (3.)\ a_n = b_n - c_n \qquad (4.)\ |a_n| = b_n + c_n.$$

From (1.) we see that $\sum_{n=0}^{\infty} b_n$ is a positive series, and since $\sum_{n=0}^{\infty} |a_n|$ converges, it follows from the comparison test that $\sum_{n=0}^{\infty} b_n$ also converges. Let its sum be B.

From (2.) we see that $\sum_{n=0}^{\infty} c_n$ is a positive series, and since $\sum_{n=0}^{\infty} |a_n|$ converges, it follows from the comparison test that $\sum_{n=0}^{\infty} c_n$ also converges. Let its sum be C.

From (3.) and the subtraction rule for series, we see that

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (b_n - c_n) = B - C.$$

In particular, $\sum_{n=0}^{\infty} a_n$ converges.